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A NOTE ON IDEALS OF OPERATORS

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ABSTRACT. An ideal $\mathcal{I}$ of $\mathcal{L}(\mathcal{H})$ is said to be multiplicatively prime if $AXB \in \mathcal{I}$ for all $X \in \mathcal{L}(\mathcal{H})$ implies A or B is in $\mathcal{I}$. The only normable multiplicatively prime ideals are $\{0\}$ and $\mathcal{K}$, the compacts. Multiplicative primeness is related to other properties an ideal may possess.

Considering the following properties that an ideal $\mathcal{I}$ of operators on separable Hilbert space may possess. (Here, ideal means two-sided and selfadjoint, but not closed.)

DEFINITION 1. An ideal $\mathcal{I}$ has the *square root property* if, for all $A \in \mathcal{I}$, we have $\sqrt{|A|} \in \mathcal{I}$. (As usual $|A| = \sqrt{A^*A}$.)

We note that $A \in \mathcal{I}$ if and only if $|A| \in \mathcal{I}$, by polar decomposition [**4**, p. 69].

DEFINITION 2. An ideal $\mathcal{I}$ is *square* if $\mathcal{I} = \mathcal{I}^2$, i.e., every element $A \in \mathcal{I}$ can be written $A = BC$ where B and C are in $\mathcal{I}$.

DEFINITION 3. An ideal $\mathcal{I}$ is *multiplicatively prime* if $AXB \in \mathcal{I}$ for all $X \in \mathcal{L}(\mathcal{H})$ implies A or B is in $\mathcal{I}$.

We remark that if $\dim \mathcal{H} \geqslant 2$, no proper two-sided ideal in $\mathcal{L}(\mathcal{H})$ is prime in the classical algebraic sense, i.e, $AB \in \mathcal{I}$ implies A or B in $\mathcal{I}$; just let

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & X \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} Y & 0 \\ 0 & B_1 \end{bmatrix}$$

where A_1, B_1 are in $\mathcal{I}$, but X and Y are not. For finite dimensions, $\mathcal{I} = \{0\}$; for infinite dimensions, it is an easy argument using s-numbers and ideal sets, cf. below, to see that X not in $\mathcal{I}$ implies $A_1 \oplus X$ not in $\mathcal{I}$.

It is evident that the improper ideal $\mathcal{L}(\mathcal{H})$ satisfies all three properties. We intend to find all the others. As a preliminary, we note that every proper two-sided ideal in $\mathcal{L}(\mathcal{H})$ is a subset of $\mathcal{K}$, the ideal of compact operators. For a compact operator T, the sequence of s-numbers of T, $s(T)$, is defined as follows: Let $\alpha_1 \geqslant \alpha_2 \geqslant \alpha_3 \geqslant \cdots$ be the eigenvalues of $|T|$; then $s(T) = \{s_n(T) = \alpha_n\} = s(|T|)$.

An *ideal set* I is a collection of sequences of real numbers $\{a_n\}_{n=1}^{\infty}$ with the following properties:

(i) if $\{a_n\} \in I$, then $a_n \geqslant 0$ for all n, and $\lim_n a_n = 0$;

(ii) if $\{a_n\} \in I$ and π is any permutation of the positive integers then $\{a_{\pi(n)}\} \in I$;

(iii) if $\{a_n\} \in I$ and $\{b_n\} \in I$, then $\{a_n + b_n\} \in I$;

(iv) if $\{a_n\} \in I$ and $0 \leqslant b_n \leqslant a_n$ for all n, then $\{b_n\} \in \mathcal{I}$.

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It is a classical result of Calkin [**2**] that there is a bijection between ideal sets and proper two-sided ideals of operators. We also have the following [**3**, Lemma 1.1], see also [**2**]:

LEMMA. *Let $\mathscr{I}$ be a proper two-sided ideal of $\mathscr{L}(\mathscr{H})$. A compact operator T belongs to $\mathscr{I}$ if and only if $s(T)$ belongs to the ideal set of $\mathscr{I}$.*

An ideal $\mathscr{I}$ is said to be *a norm ideal* if there is a norm $\|\cdot\|_{\mathscr{I}}$ on $\mathscr{I}$ with the following properties:

(i) $(\mathscr{I}, \|\cdot\|_{\mathscr{I}})$ is a Banach space;

(ii) $\|STR\|_{\mathscr{I}} \leqslant \|S\|\,\|T\|_{\mathscr{I}}\|R\|$ for all $R, S \in \mathscr{L}(\mathscr{H})$, for all $T \in \mathscr{I}$;

(iii) $\|T\|_{\mathscr{I}} = \|T\|$ for T of rank one.

The canonical examples of norm ideals are the Schatten ideals C_p, $1 \leqslant p \leqslant \infty$, where $\|T\|_p$ is the l^p norm of $s(T)$.

THEOREM 1. *Consider the following properties of an ideal $\mathscr{I}$:*

(1) *$\mathscr{I}$ is multiplicatively prime*;

(2) *$\mathscr{I}$ has the square root property*;

(3) *$\mathscr{I}$ is square.*

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3).

PROOF. (1) $\Rightarrow$ (2). Suppose $A \in \mathscr{I}$; hence $|A| \in \mathscr{I}$. Look at the maping $X \to \sqrt{|A|}\,X\sqrt{|A|}$ for $X \in \mathscr{L}(\mathscr{H})$. We claim the range of this mapping is in $\mathscr{I}$; if so, then by the property of being multiplicatively prime, we have $\sqrt{|A|} \in \mathscr{I}$.

But $X \to CXD$ has range in $\mathscr{I}$ if and only if $s(C)s(D)$ belongs to I, the ideal set of $\mathscr{I}$, by [**3**, Lemma 5.4 and 5.5]. Then $s(\sqrt{|A|})s(\sqrt{|A|}) = s(|A|) \in I$ by our earlier lemma, completing the proof.

(2) $\Rightarrow$ (3). The polar decomposition writes $A = U|A| = (U\sqrt{|A|})\sqrt{|A|}$ as a product of elements in $\mathscr{I}$.

(3) $\Rightarrow$ (2). This follows from Theorem 2.1 of [**6**].

REMARK. Let $\mathscr{I}_p = \bigcup_{K \geqslant 0} C_{p2^K}$ where $p \geqslant 1$; then $\mathscr{I}$ has the square root property, for if $\Sigma|\beta_i|^r < \infty$, then $\Sigma(\sqrt{|\beta_i|})^{2r} < \infty$; thus if $A \in C_r \subseteq \mathscr{I}_p$ then $\sqrt{|A|} \in C_{2r} \subseteq \mathscr{I}_p$.

On the other hand, $\mathscr{I}$ is not multiplicatively prime, since there are compact operators A and B such that $A, B \notin C_p$ for every p, $1 \leqslant p < \infty$, but such that $X \to AXB$ takes values in C_1 [**3**, Example 5.8].

PROPOSITION 2. *The ideals $0, \mathscr{F}$, and $\mathscr{K}$ are multiplicatively prime.*

PROOF. For $\mathscr{K}$, this is Proposition 4.1 of [**3**].

If $AXB = 0$ for all X with $A, B \neq 0$, let u be so that $Bu = v \neq 0$; and let w be so that $Aw \neq 0$. Then for X the rank-one operator $X(e) = \langle e, v\rangle w$, we have $(AXB)(u) \neq 0$.

If AXB is finite rank for all X, but A, B are not finite rank, choose $\{e_i\}_{i=1}^{\infty}$ such that $\{Be_i\}_{i=1}^{\infty}$ is an orthonormal basis for the range of B; and choose $\{f_i\}_{i=1}^{\infty}$ an orthonormal set so that $\{Af_i\}$ is a basis for the range of A. Then for the partial isometry X: $Be_i \to f_i$, we have AXB is not finite rank.

THEOREM 3. *If $\mathscr{I}$ is a proper norm ideal different from $\mathscr{L}(\mathscr{H})$, the following are equivalent*:

(1) *$\mathscr{I}$ is multiplicatively prime*,

(2) *$\mathscr{I}$ has the square root property*,

(3) *$\mathscr{I}$ is square*,

(4) *$\mathscr{I} = \{0\}$ or $\mathscr{I} = \mathscr{K}$.*

PROOF. By Theorem 1 and Proposition 2, the only implication we need to prove is (3) $\Rightarrow$ (4). But this is precisely the content of [**7**, Theorem 2.9].

REMARKS. The above results lend to the following question: Which ideals $\mathscr{I}$ of $\mathscr{L}(\mathscr{H})$ are multiplicatively prime?

To see if $\mathscr{I}$ is multiplicatively prime, we ask whether $AXB \in \mathscr{I}$ for all $X \in \mathscr{L}(\mathscr{H})$ implies that A or B is in $\mathscr{I}$.

From [**3**, Lemma 5.1 and Corollary 5.2], we obtain the facts that if either A or B is not compact the only way for AXB to be in $\mathscr{I}$ for all X in $\mathscr{L}(\mathscr{H})$ is for B (for A respectively) to be in $\mathscr{I}$.

From Theorem 3, we see that norm ideals cannot be square, and thus cannot be multiplicatively prime. This also follows from Theorem 7.11 in [**7**], which states that if $\mathscr{I} \subsetneq \mathscr{K}$ is any norm ideal, there are ideals $\mathscr{I}_1, \mathscr{I}_2$ with $\mathscr{I} \subsetneq \mathscr{I}_k$ and $\mathscr{I} = \mathscr{I}_1 \cap \mathscr{I}_2$. If we choose $A_k \in (\mathscr{I}_k/\mathscr{I})$, then we have for all X in $\mathscr{L}(\mathscr{H})$ that $A_1XA_2 \in \mathscr{I}_1 \cap \mathscr{I}_2 = \mathscr{I}$. From [**3**, 5.4] we have $s(A_1)s(A_2) \in I$, but $s(A_i) \notin I$ since $A_i \notin \mathscr{I}$.

We conjecture that, among nonnorm ideals, only $\mathscr{F}$ is multiplicatively prime.

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