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AN ERROR ESTIMATE FOR CONTINUED FRACTIONS

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ABSTRACT. New and improved truncation error bounds are derived for continued fractions $K(a_n/1)$, where $a_n \to 0$. The geometrical approach is somewhat unusual in that it involves both isometric circles and fixed points of bilinear transformations.

Continued fractions of the form

$$\frac{a_1}{1} + \frac{a_2}{1} + \cdots, \tag{1}$$

where $a_n \in \mathbf{C}$, $a_n \neq 0$ for $n \geqslant 1$, and $a_n \to a$, are called limit periodic. In this paper we shall obtain new and improved a priori error bounds for both the classical continued fraction (1) and its modified form [**2**] in the case $a = 0$.

Recently obtained results in error analysis/acceleration by Thron and Waadeland [**6, 7**] apply only in the case $a \neq 0$, and the author's results along these lines [**2**] require the sort of bounds described in the present paper.

(1) may be perceived as a composition of bilinear transformations in the following way: Set $t_n(z) = a_n/(1+z)$ and $T_n(z) = t_1 \circ \cdots \circ t_n(z)$ for $n \geqslant 1$. The nth approximant of (1) is then $T_n(0)$.

The isometric circle of $t_n(z)$ is defined as $I_n = \{z\colon |z+1| = \sqrt{|a_n|}\}$, $n \geqslant 1$. By an application of t_n, distances outside I_n are diminished, and those inside I_n are increased [**1**]. t_n operating on a circle exterior to I_n contracts the circle through an inversion in I_n and reflects and rotates the resulting circle. If t_n has an attractive fixed point α_n, and α_n lies inside the original circle, it also lies inside the transformed circle.

Let us assume that $|\arg(a_n + 1/4)| < \pi$ and $\operatorname{Re}(\sqrt{a_n + 1/4}) > 0$. Then $\alpha_n = -1/2 + \sqrt{a_n + 1/4}$ is the *attractive* fixed point of t_n (i.e., t_n has two distinct fixed points α_n and β_n and $|\alpha_n| < |\beta_n|$). We write

$$A_n = \sup_{m \geqslant n} \sqrt{|a_m|}, \qquad A = A_1,$$

$$P_n = \frac{2-A}{2 - A - 2A_n^2/A}, \quad \text{and} \quad \varepsilon_n = P_n \cdot \sup_{m \geqslant n} \left| -\frac{1}{2} + \sqrt{a_m + \tfrac{1}{4}} \right|$$

for $n \geqslant 1$.

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With reference to (1) we have

THEOREM. *If*
(i) $a_n \to 0$,
(ii) $A < 2/3$, *and*
(iii) $\sup_{m \geqslant n} |-\frac{1}{2} + \sqrt{a_m + \frac{1}{4}}| < (1 - A)/P_1$, $n \geqslant 1$ *are satisfied,*
then (1) *converges to a value* T, *and*

$$|T_n(\mu_n) - T| < 2\varepsilon_n \prod_{m=1}^{n} \left(\frac{A_m}{1 - \varepsilon_m} \right)^2, \qquad n \geqslant 1, \textit{ where } \mu_n = 0 \textit{ or } \mu_n = \alpha_{n+1}$$

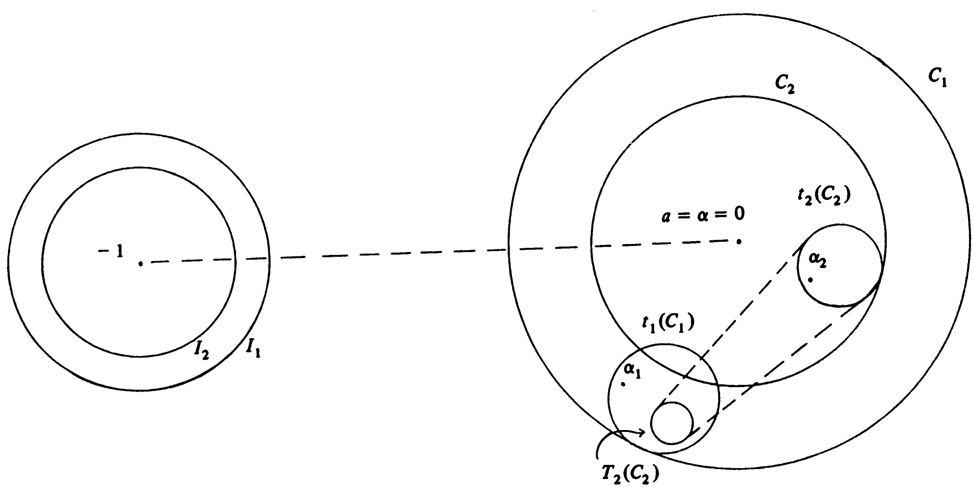
$$= \left| -\tfrac{1}{2} + \sqrt{a_{n+1} + \tfrac{1}{4}} \right|.$$

PROOF. The condition $A < 2/3$ implies $\sqrt{|a_n|} < 1$ for $n \geqslant 1$. Thus $a = \alpha = 0$ lies outside I_n for $n \geqslant 1$. We seek disks $C_n = \{z\colon |z| \leqslant R_n\}$ such that $C_{n+1} \subset C_n$, $t_m(C_n) \subseteq C_n$ for $m \geqslant n$, C_n lies outside I_n, α_m lies inside C_n for $m \geqslant n$, and $\mathrm{Rad}[T_n(C_n)] \to 0$. With the notation

$$M_n = \sup_{m \geqslant n} |\alpha_m| \quad \text{and} \quad Y_n = \sup_{m \geqslant n} \{ s_m\colon s_m = \mathrm{Rad}[t_m(C_n)] \},$$

the following two conditions reflect these criteria:
I. $R_n \geqslant M_n + 2Y_n$, and
II. $R_n < 1 - A \leqslant 1 - A_n$ for $n \geqslant 1$ (see the Figure).
The transformation $t_m(C_n)$ involves an inversion in I_m, so that

$$s_m = \left(\sqrt{|a_m|}\right)^2 / [(1 - R_n)(1 + R_n)], \qquad m \geqslant n.$$



This implies $Y_n = [A_n^2/(1 - R_n)^2] \cdot R_n$. Assuming for the moment that $R_n < 1 - A$, we find that $Y_n < [A_n^2/(2A - A^2)] \cdot R_n$. Therefore, we define R_n in the following way:

$$R_n := M_n + 2A_n^2 R_n/(2A - A^2) > M_n + 2Y_n,$$

thus insuring I. This gives $R_n = M_n \cdot P_n$, where $P_n \leqslant (2 - A)/(2 - 3A) = P_1$.

Condition II is guaranteed by requiring

$$M_n < (1 - A)/P_1 \leqslant (1 - A)/P_n, \qquad n \geqslant 1.$$

It follows that R_n $(\downarrow)$ to 0, since M_n $(\downarrow)$ to 0 and $P_n > 0$ decreases monotonically. If C is a circle of radius Y contained in C_n, then the radius Y^* of the transformed circle $t_n(C_n)$ satisfies

$$Y^* < \left[\frac{\sqrt{|a_n|}}{1 - R_n}\right]^2 \cdot Y \leqslant \frac{A_n^2}{(1 - R_n)^2} \cdot Y.$$

Therefore,

$$\operatorname{Rad}[t_n(C_n)] < \left(\frac{A_n}{1 - R_n}\right)^2 \cdot R_n,$$

$$\operatorname{Rad}[t_{n-1} \circ t_n(C_n)] < R_n \prod_{m=n-1}^{n} \left(\frac{A_m}{1 - R_m}\right)^2,$$

$$\vdots$$

$$\operatorname{Rad}[T_n(C_n)] < R_n \prod_{m=1}^{n} \left(\frac{A_m}{1 - R_m}\right)^2,$$

where $\lim_{m \to \infty} [A_m/(1 - R_m)] = 0$.

Now, since $T_n(0)$, $T_n(\alpha_{n+1}) \in T_n(C_n)$ and $\lim T_n(\mu_n) = T$ if $\mu_n \to 0$ [**4**], we obtain the conclusion of the theorem.

EXAMPLE.

$$\frac{11/10^2}{1} + \frac{21/20^2}{1} + \cdots + \frac{(10n + 1)/(10n)^2}{1} + \cdots.$$

Bounds for $|T_n(0) - T|$ are

n	Actual Error	Theorem est.	Theorem 1 [**3**] est.	Theorem 2.1 [**5**] est.
1	5.3×10^{-3}	5.3×10^{-2}	N/A	N/A
3	4.1×10^{-6}	4.2×10^{-5}	5.4×10^{-3}	3.5×10^{-3}
5	1.2×10^{-9}	1.5×10^{-8}	7.1×10^{-6}	2.0×10^{-4}

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