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HOPF FORMULA AND MULTITIME HAMILTON-JACOBI EQUATIONS

P. L. LIONS AND J-C. ROCHET

ABSTRACT. Problems arising in mathematical economics lead to the study of multitime Hamilton-Jacobi equations. Using commutation properties of the semigroups for the standard equation, we propose a generalization of the Hopf formula that gives explicit solutions of these equations.

I. Hopf formula. We consider the following Cauchy problem for the Hamilton-Jacobi equation:

$$(1)\qquad \begin{aligned} &\partial_u/\partial t + H(Du) = 0 && \text{in } \mathbf{R}^N \times [0,T], \\ &u(x,0) = u_0(x) && \text{in } \mathbf{R}^N. \end{aligned}$$

M. G. Crandall and P. L. Lions [**2, 3**] introduced the notion of viscosity solution of (1): In particular, they proved the existence and uniqueness of a viscosity solution of (1) in BUC($\mathbf{R}^N \times [0,T]$) when H is continuous on $\mathbf{R}^N$ and u_0 belongs to BUC($\mathbf{R}^N$), the set of functions bounded and uniformly continuous on $\mathbf{R}^N$. This solution is denoted by $(S_H(t)u_0)(x)$. Of course, the family $(S_H(t))_{t\geqslant 0}$ defines a strongly continuous semigroup on BUC($\mathbf{R}^N$).

When u_0 is in BUC($\mathbf{R}^N$) and H is convex and coercive (i.e., $\lim_{|p|\to+\infty} H(p)/|p| = +\infty$), this viscosity solution is given explicitly by the Oleinik-Lax formula (cf. [**7**])

$$(2)\qquad u(x,t) = \operatorname*{Inf}_{y\in\mathbf{R}^N} \operatorname*{Sup}_{z\in\mathbf{R}^N} \{u_0(y) + (z, x-y) - tH(z)\}.$$

A proof of this fact may be found in Lions [**8**]. This can also be written

$$(3)\qquad u(x,t) = \operatorname*{Inf}_{y\in\mathbf{R}^N} \left\{u_0(y) + tH^*\left(\frac{x-y}{t}\right)\right\}$$

where H^* is the Fenchel conjugate of H:

$$(4)\qquad H^*(p) = \operatorname*{Sup}_{z\in\mathbf{R}^N} \{(p,z) - H(z)\}.$$

On the other hand, when u_0 is convex and H is continuous, Hopf [**5**] proposed the following formula (dual of (2)):

$$(5)\qquad v(x,t) = \operatorname*{Sup}_{z\in\mathbf{R}^N} \operatorname*{Inf}_{y\in\mathbf{R}^N} \{u_0(y) + (z, x-y) - tH(z)\}.$$

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This can also be written

$$v(x,t) = \operatorname*{Sup}_{z \in \mathbf{R}^N} \{(x,z) - u_0^*(z) - tH(z)\} \tag{6}$$

or

$$v(x,t) = (u_0^* + tH)^*(x). \tag{7}$$

The Hopf formula defines a convex function, which is a solution of (1) for a.e. (x,t) in its domain. This domain will be all of $\mathbf{R}^N$, provided that the following property is fulfilled:

$$\lim_{|p| \to +\infty} \frac{u_0^*(p) + tH(p)}{|p|} = +\infty, \quad \text{uniformly for } t \text{ in } [0,T]. \tag{8}$$

The following shows that v is a viscosity solution of (1): This fact (under slightly less general assumptions) was proved in Bardi and Evans [**1**], but our method is simpler and more direct.

PROPOSITION 1. *Under assumption* (8), *if* u_0 *is convex and* u_0, H *are continuous, the Hopf formula defines a viscosity solution of* (1).

REMARK. Of course, if $u_0 \in \mathrm{UC}(\mathbf{R}^N)$, u_0 is in fact Lipschitz on $\mathbf{R}^N$, and, thus, v given by (7), satisfies $D_x v \in L^\infty(\mathbf{R}^N \times]0,T[)$. Then v is the unique viscosity solution with this regularity in view of [**4**, **6**, and **2**].

Proposition 1 is a consequence of the following lemma, which ensures that the Hopf formula defines a semigroup.

LEMMA 1. *For any functions* u_0, H *from* $\mathbf{R}^N$ *to* $\mathbf{R}$ *and any positive numbers* t, s *we have*

$$\left((u_0^* + tH)^{**} + sH\right)^* = \left(u_0^* + (t+s)H\right)^*. \tag{9}$$

PROOF OF LEMMA 1. Since for all u, $u^{**} \leqslant u$, and since Fenchel's transformation is order-reversing, we have

$$(u_0^* + tH)^{**} + sH \leqslant u_0^* + (t+s)H$$

and

$$\left((u_0^* + tH)^{**} + sH\right)^* \geqslant \left(u_0^* + (t+s)H\right)^*.$$

For the other inequality let us remark that

$$\frac{s}{t+s} u_0^* + \frac{t}{t+s}\left(u_0^* + (t+s)H\right)^{**} \leqslant u_0^* + tH.$$

Since the left side is convex, we get

$$\frac{s}{t+s} u_0^* + \frac{t}{t+s}\left(u_0^* + (t+s)H\right)^{**} \leqslant (u_0^* + tH)^{**}.$$

Thus

$$\left(u_0^* + (t+s)H\right)^{**} - (u_0^* + tH)^{**} \leqslant (s/t)\left[(u_0^* + tH)^{**} - u_0^*\right] \leqslant sH.$$

Consequently,

$$\left(u_0^* + (t+s)H\right)^{**} \leqslant \left(u_0^* + tH\right)^{**} + sH.$$

Taking conjugates, we obtain our second inequality. □

PROOF OF PROPOSITION 1. We already know that (7) defines a convex function satisfying (1) at each point of differentiability. In particular, it is a viscosity subsolution of (1) for if v is superdifferentiable at (x, t) then it is also differentiable and, thus,

$$(\partial v/\partial t)(x,t) + H(Dv(x,t)) = 0$$

(this is in fact a very special case of arguments given in Lions [**8**]). Let us now prove that v is also a viscosity supersolution. In other words, we have to show that, for any (x_0, t_0) in $\mathbf{R}^N \times [0, T]$ and (p, q) in the subdifferential of v at (x_0, t_0), we have

$$q + H(p) \geqslant 0.$$

To prove this inequality, we adapt the arguments of Lions and Nisio [**9**]. By convexity and the definition of the subdifferential of v, we have

(10) $\quad \forall (x,t) \in \mathbf{R}^N \times [0,T], \qquad v(x,t) \geqslant v(x_0,t_0) + (p, x - x_0) + q(t - t_0).$

Let us denote by $\bar{S}_H(t)$ the semigroup defined by (8):

$$\bar{S}_H(t)u_0 = \left(u_0^* + tH\right)^*.$$

By Lemma 1 we have for any s in $[0, t_0]$,

$$v(x_0,t_0) = \left[\bar{S}_H(s)v(\cdot, t_0 - s)\right](x_0).$$

Since $\bar{S}_H(s)$ is order preserving, we get, by (10),

$$v(x_0,t_0) \geqslant \left(\bar{S}_H(s)\varphi\right)(x_0),$$

where $\varphi(x) = v(x_0,t_0) + (p, x - x_0) - sq$. Thus

$$v(x_0,t_0) \geqslant \sup_{z \in \mathbf{R}^N} \left\{(z, x_0) - t_0 H(z) - \varphi^*(z)\right\}.$$

But

$$\varphi^*(z) = \begin{cases} sq - v(x_0,t_0) + px_0 & \text{if } z = p, \\ +\infty & \text{if } z \neq p. \end{cases}$$

Consequently, $v(x_0,t_0) \geqslant -sq - t_0H(p) + v(x_0,t_0)$. Thus, for any s in $[0, t_0]$ we get

$$sq + t_0 H(p) \geqslant 0,$$

which implies, in particular, that $q + H(p) \geqslant 0$. □

REMARK. If we consider the natural extension of (1)

$$(1') \qquad \begin{aligned} &\partial u/\partial t + H(t, Du) = 0 && \text{in } \mathbf{R}^N \times]0,T[, \\ &u(x,0) = u_0(x) && \text{in } \mathbf{R}^N, \end{aligned}$$

then a natural extension of the Hopf formula is

$$(6') \qquad v(x,t) = \sup_{z \in \mathbf{R}^N} \left\{(x,z) - u_0^*(z) - \int_0^t H(s,z)\,ds\right\}$$

or

$$(7') \qquad v(x,t) = \left(u_0^* + \int_0^t H(s,\cdot)\,ds\right)^*(x).$$

But (6′), (7′) *do not* define a viscosity solution of (1); indeed, if this were the case, this formula would define an evolution operator and, by a density argument ($u_0^* \to 0$), we would have

$$\left\{\left(\int_0^t H(\lambda,\cdot)\,d\lambda\right)^{**} + \int_t^{t+s} H(\lambda,\cdot)\,d\lambda\right\}^* = \left(\int_0^{t+s} H(\lambda,\cdot)\,d\lambda\right)^*$$

for all t, $s \geqslant 0$, and this is, in general, false! □

II. Commutation of the semigroups. The Hopf formula sheds some light on a new property of commutation of the semigroups:

PROPOSITION 2. *If u_0, H_1, H_2 are convex, continuous, and such that* (8) *holds for H_1 and H_2, then we have for all positive t, s,*

$$(11) \qquad S_{H_1}(t)S_{H_2}(s)u_0 = S_{H_2}(s)S_{H_1}(t)u_0 = S_{tH_1+sH_2}(1)u_0.$$

PROOF. $S_{H_1}(t)S_{H_2}(s)u_0 = [(u_0^* + sH_2)^{**} + tH_1]^*$.
If H_2 is convex this is equal to

$$S_{tH_1+sH_2}(1)u_0 = \left[u_0^* + tH_1 + sH_2\right]^*. \quad \square$$

It is easy to find a counterexample to (11) if H_1 and H_2 are not convex. On the other hand, a reexamination of the Lax formula shows that the commutation property can also be proved for u_0 in BUC($\mathbf{R}^N$) and convex continuous H_1 and H_2.

PROPOSITION 3. *If $u_0 \in$ BUC($\mathbf{R}^N$) and H_1, H_2 are convex, then* (11) *holds for any positive t, s.*

PROOF. By the Oleinik-Lax formula we have

$$\left(S_{H_1}(t)S_{H_2}(s)u_0\right)(x) = \operatorname*{Inf}_{z\in\mathbf{R}^N}\ \operatorname*{Inf}_{y\in\mathbf{R}^N}\left\{u_0(y) + (tH_1)^*(x-z) + (sH_2)^*(z-y)\right\}.$$

But

$$\operatorname*{Inf}_{z\in\mathbf{R}^N}\left\{(tH_1)^*(x-z) + (sH_2)^*(z-y)\right\} = (tH_1 + sH_2)^*(x-y).$$

Thus

$$\left(S_{H_1}(t)S_{H_2}(s)u_0\right)(x) = \left(S_{tH_1+sH_2}(1)u_0\right)(x). \quad \square$$

REMARK. The same commutation property obviously holds for small t, if u_0, H_1, H_2 are smooth; indeed by the method of characteristics (see, for example, [**8**]) we have the following: Let $v = S_{tH_1+sH_2}(1)u_0(x)$. Then for t, s small there exists a unique x_1 in $\mathbf{R}^N$ such that

$$x_1 + tH_1'(Du_0(x_1)) + sH_2'(Du_0(x_1)) = x, \qquad Du_0(x_1) = D_x v(x,t,s),$$
$$v(x,t,s) = u_0(x_1) + t\left\{H_1'\cdot Du_0(x_1) - H_1\right\} + s\left\{H_2'\cdot Du_0(x_1) - H_2\right\}.$$

Next if we set $x_2 = x_1 + tH_1'(Du_0(x_1))$, we have

$$Du_0(x_1) = D_x\big(S_{H_1}(t)u_0\big)(x_2),\ S_{H_1}(t)u_0(x_2) = u_0(x_1) + t\big\{H_1' \cdot Du_0(x_1) - H_1\big\},$$

and observing that $x = x_2 + sH_2'(D_x(S_{H_1}(t)u_0)(x_2))$, we conclude that

$$v(x,t,s) = S_{H_2}(s)S_{H_1}(t)u_0(x) \quad \text{for small } t,\ s \geqslant 0.$$

Still for small t, s, the same property would hold for general Hamiltonians $H_1(x, p)$, $H_2(x, p)$, provided we have

$$\frac{\partial H_1}{\partial p} \cdot \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial x} \cdot \frac{\partial H_2}{\partial p} = 0 \quad \text{on } \mathbf{R}^N \times \mathbf{R}^N;$$

i.e., $[H_1, H_2] = 0$ (H_1, H_2 are in involution)!

III. Multitime equations. Problems arising in mathematical economics [**10**] lead to the following Cauchy problem for what we call (by analogy) the multitime Hamilton-Jacobi equation:

$$(12)\qquad \begin{aligned} &\partial u/\partial t + H_1(Du) = 0 && \text{in } \mathbf{R}^N \times [0,T]^2,\\ &\partial u/\partial s + H_2(Du) = 0 && \text{in } \mathbf{R}^N \times [0,T]^2,\\ &u(x,0,0) = u_0(x) && \text{in } \mathbf{R}^N. \end{aligned}$$

Notice that (12) is apparently an overdetermined system of p.d.e.'s.

As a consequence of Propositions 2 and 3, we obtain explicit formulae giving weak solutions of (12).

PROPOSITION 4. *If u_0 is convex on $\mathbf{R}^N$, u_0, H_1, H_2 are continuous and if* (8) *holds for H_1 and H_2 then formula* (13) *defines a convex function v on $\mathbf{R}^N \times [0,T]^2$, solving* (12) *a.e.*

$$(13)\qquad v(x,t,s) = \big(u_0^* + tH_1 + sH_2\big)^*.$$

PROPOSITION 5. *If $u_0 \in \mathrm{BUC}(\mathbf{R}^N)$, H_1, H_2 are convex continuous, and if either $Du_0 \in L^\infty$, or H_1, H_2 are coercive, then*

$$V = S_{H_1}(t)S_{H_2}(s)u_0 = S_{H_2}(s)S_{H_1}(t)u_0 = S_{tH_1+sH_2}(1)u_0$$

is Lipschitz on $\mathbf{R}^N \times [\varepsilon, T]^2$ for any $\varepsilon > 0$ and solves (12) *a.e.*

REMARKS. (i) In Proposition 4, for fixed $t > 0$, v is *not*, in general, a viscosity solution of $\partial u/\partial s + H_2(Du) = 0$, although this is true in Proposition 5. However, in both cases $v = S_{tH_1+sH_2}(1)u_0$, and thus v is a viscosity solution on each half-line connecting $(0,0)$ with (t,s).

(ii) If u_0, H_1, H_2 are smooth (say $W^{2,\infty}(\mathbf{R}^N)$), then we may apply the method of characteristics (cf. Remark in §II), which yields that $v = S_{H_1}(t)S_{H_2}(s)u_0$ is a smooth solution of (12) provided T is small ($T < T_0(u_0, H_1, H_2)$).

(iii) Except for these special cases, we do not know if (12) admits global solutions for every H_1, H_2, u_0: A good tentative solution could be $S_{tH_1+sH_2}(1)u_0$, but we are unable to decide if it solves (12) a.e.

(iv) Of course, Propositions 4–5 extend to an arbitrary number of times—that is, m equations in $\mathbf{R}^N \times [0,T]^m$ involving m different Hamiltonians for a *single* unknown function.

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CEREMADE, PARIS IX UNIVERSITY, PLACE DE LATTRE DE TASSIGNY, 75775 PARIS CEDEX 16, FRANCE