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A FINITE BUT NOT STABLY FINITE C^*-ALGEBRA

N. P. CLARKE[1]

ABSTRACT. In E. G. Effros's paper [**4**] given at the 1980 Kingston Conference of the American Mathematical Society, he stated that "an example of a finite but not stably finite C^*-algebra has yet to be found." This paper seeks to give an example of such an algebra by using a simple application of the duality between K- and Ext-theory arising from the work of Brown, Douglas and Fillmore (see, for example, [**2** and **3**]).

Throughout this paper the C^*-algebras will be separable and unital, except in the obvious cases such as $\mathscr{K}$, the algebra of compact operators on an infinite-dimensional, separable Hilbert space $\mathscr{H}$. $\mathscr{L}$ will denote the algebra of all bounded, linear operators on $\mathscr{H}$, so that $\mathscr{K}$ is the only ideal (= closed, two-sided ideal) of $\mathscr{L}$. The quotient is $\mathscr{Q}$, the Calkin algebra. Denote by $U(\mathscr{H})$ the group of all unitary operators in $\mathscr{L}$. Finally, for each positive integer n let M_n be the algebra of complex valued, $n \times n$ matrices.

Let A be a separable, unital C^*-algebra. We say that A is *finite* if for every pair of elements x, y in A,

$$xy = 1 \quad \text{implies} \quad yx = 1.$$

A is *stably finite* if every finite-dimensional matrix algebra over A is finite; that is, if $M_n \otimes A$ is finite for each positive integer n.

It is now time to introduce the notion of K-theory. Again, let A be a separable, unital C^*-algebra. We can define a semigroup $J(A)$ as the set of projections in $\mathscr{K} \otimes A$ with direct sum as addition, modulo equivalence of projections. Following [**6**] this equivalence of projections can be defined in three different ways (equivalence, $*$-equivalence and unitary equivalence), but as Kaplansky shows [**6**, Chapter 3, in particular Theorem 28], these all give the same relation on a stable algebra. Alternatively, $J(A)$ can be defined as equivalence classes of idempotents in $\mathscr{K} \otimes A$ with direct sum as addition. Again, the two relevant types of equivalence (equivalence and invertible equivalence) turn out to be the same, and this version of $J(A)$ is the same as the last [**6**, Theorem 27]. For e an idempotent in $\mathscr{K} \otimes A$, denote by $[e]_J$ the class of e in $J(A)$. Following J. L. Taylor [**9**, Chapter 6], let $K_0(A)$ be the

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universal group of the semigroup $J(A)$ and let $[e]_0$ denote the element of $K_0(A)$ induced by the element $[e]_J$ of $J(A)$.

The definition of $K_1(A)$ is similar, and the reference is once again [**9**]. For x an invertible element in some $M_n \otimes A$, let $[x]_1$ denote the class of x in $K_1(A)$.

The construction of the finite but not stably finite C^*-algebra will be carried out within the framework of the work of Brown, Douglas and Fillmore [**2** or **3**]. Thus we shall consider essential, unital extensions. The construction uses a little trick which would apply in the same way within the framework of Kasparov [**7**] (see J. Rosenberg's paper [**8**] for an account of how the different frameworks fit together).

The relevant theorem is the following [**1**, Theorem 2]:

THEOREM (UNIVERSAL COEFFICIENT THEOREM). *There is a natural short exact sequence*

$$0 \to \operatorname{ext}^1_{\mathbb{Z}}(K_0(A), \mathbb{Z}) \to \operatorname{Ext}(A) \xrightarrow{\gamma} \operatorname{Hom}(K_1(A), \mathbb{Z}) \to 0$$

which splits noncanonically.

Here A is a unital, separable, and also commutative C^*-algebra, and we know [**3**] that $K_*(A) \simeq K^*(\hat{A})$. The action of γ is via the index map, as will be seen later.

Now let $A = C(T^3)$ denote the commutative C^*-algebra of complex-valued, continuous functions on the 3-torus T^3. A is generated by three commuting unitary elements, u_1, u_2 and u_3, each with spectrum the whole of the circle T. Any element of $U(A)$, the unitary group of A, is homotopic in $U(A)$ to $u_1^{j_1} \cdot u_2^{j_2} \cdot u_3^{j_3}$ for some integers j_1, j_2 and j_3. This gives us three generators for $K_1(A)$, namely, $[u_1]_1$, $[u_2]_1$ and $[u_3]_1$. However, $K_1(A) \simeq K^1(T^3) \simeq \Lambda^{\text{odd}}(\mathbb{Z}^3) \simeq \mathbb{Z}^4$, so there must be a fourth generator of $K_1(A)$ with no representative in A. Let v in $M_n \otimes A$ be a unitary representative of this last generator. (Note that there is a continuous map from T^3 to S^3 given by regarding T^3 as $\mathbb{R}^3/\mathbb{Z}^3$ and identifying the faces to a point. This gives a group homomorphism: $K^1(S^3) \simeq \mathbb{Z} \to K^1(T^3)$ of rank one, and it takes the standard generator of $K^1(S^3)$ into a fourth generator for $K^1(T^3)$, say $[v]_1$. Thus we could take $n = 2$.)

$K_0(A)$ and $K_1(A)$ are free $\mathbb{Z}$-modules, so the first term in the Universal Coefficient Theorem vanishes [**5**, Theorem 6.6] and we have an isomorphism

$$\gamma\colon \operatorname{Ext}(A) \xrightarrow{\simeq} \operatorname{Hom}(K_1(A), \mathbb{Z}) \simeq \mathbb{Z}^4.$$

Therefore, we can pick an element τ of $\operatorname{Ext}(A)$ satisfying

$$\gamma(\tau)([u_i]_1) = 0, \quad i = 1, 2, 3; \qquad \gamma(\tau)([v]_1) = 1.$$

Let the exact sequence corresponding to τ be

$$0 \to \mathcal{K} \xrightarrow{\varphi} E \xrightarrow{\pi} A \to 0.$$

We shall regard matrix algebras over E as being embedded in $\mathscr{L}$, and matrix algebras over A as being embedded in $\mathscr{Q}$, so π can also be the canonical map: $\mathscr{L} \to \mathscr{Q}$. The exact sequence gives rise to a six-term exact loop

$$(\dagger)\qquad \begin{array}{ccccccc} \mathbb{Z} \simeq & K_0(\mathscr{K}) & \overset{\varphi_0}{\to} & K_0(E) & \overset{\pi_0}{\to} & K_0(A) \\ & \delta_1 \uparrow & & & & \downarrow \delta_0 \\ & K_1(A) & \underset{\pi_1}{\leftarrow} & K_1(E) & \underset{\varphi_1}{\leftarrow} & K_1(\mathscr{K}) = 0 \end{array}$$

where the action of $\gamma(\tau)$ on $K_0(A)$ and $K_1(A)$ is given by δ_0 and δ_1 [**2**, §4]. That is,

$$\delta_1([u_i]_1) = 0, \quad i = 1, 2, 3; \qquad \delta_1([v]_1) = 1.$$

Looking at diagram ($\dagger$), it is clear that δ_1 is onto, so φ_0 is the zero map and π_0 is an isomorphism. Using the usual construction for δ_1 (see, for example, [**9**]), we get

$$\delta_1([v]_1) = [p]_0,$$

where p can be regarded as a nonzero projection in $M_{2n} \otimes \mathscr{K}$ and $\pi\varphi p = 0$. We know that $\varphi_0([p]_0) = 0$, which is really saying that $[p]_0 = [0]_0$ in $K_0(E)$ since we can regard $\mathscr{K}$ as being embedded in E by φ. Thus, for some positive k there is a projection r in $M_k \otimes E$ with

$$[p]_J \oplus [r]_J = [0_{2n}]_J \oplus [r]_J,$$

where equality is in $J(E)$. By increasing n if necessary and choosing k large enough, we may assume that $k = 2n$. Dropping the subscript J we have,

$$[p] \oplus [r] \oplus [1_k - r] = [0_k] \oplus [r] \oplus [1_k - r],$$

so,

$$[p] \oplus [1_k] = [0_k] \oplus [1_k].$$

We can now find a, b, c and d in $M_k \otimes E$ with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} 0_k & 0_k \\ 0_k & 1_k \end{pmatrix},$$

$$\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} p & 0_k \\ 0_k & 1_k \end{pmatrix}.$$

Thus,

$$aa^* + bb^* = 0_k, \quad \text{so} \quad a = b = 0_k;$$
$$cc^* + dd^* = 1_k;$$
$$c^*c = p \neq 0_k; \qquad d^*d = 1_k.$$

Therefore, d is a nonunitary isometry in $M_k \otimes E$, and E is not stably finite.

We must now show that E is finite. Suppose not. Then we can find x, y in E with $xy = 1$ but $yx \neq 1$. Since A is finite, we know that $yx = 1 - e$, where e is an idempotent in $\mathscr{K}$. In particular, x is Fredholm. A simple calculation shows that

$$\operatorname{Index}(x) = \operatorname{rank}(e) > 0.$$

However, πx induces an element in $K_1(A)$ which lies within the subgroup generated by $[u_1]_1$, $[u_2]_1$ and $[u_3]_1$, so $\delta_1([\pi x]_1) = 0$. This is a contradiction, since by, for example, [**2**, §4], $\mathrm{Index}(x) = \delta_1([\pi x]_1)$.

REMARKS. 1. The example given here of a finite but not stably finite C^*-algebra is far from being simple, although it is primitive. To my knowledge, a simple example has not yet been found.

2. Given any separable, nuclear, unital, finite C^*-algebra A with a torsion-free part of its K_1-group not represented in $U(A)$ the above construction will generate a finite but not stably finite E, since the Universal Coefficient Theorem, etc., goes through to the noncommutative case (see [**8**]). This method will never result in a simple E, but it may be possible to start with a simple A, thus making E as simple as possible. Such an A may, however, not exist.

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DEPARTMENT OF MATHEMATICS, J. C. MAXWELL BUILDING, THE KING'S BUILDINGS, MAYFIELD ROAD, EDINBURGH EH9 3JZ, SCOTLAND