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NONAMENABILITY AND BOREL PARADOXICAL DECOMPOSITIONS FOR LOCALLY COMPACT GROUPS

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ABSTRACT. We show that a locally compact group G is not amenable if and only if it admits a Borel paradoxical decomposition.

In 1938 A. Tarski [7] proved the following remarkable theorem. *Let G be a group acting invertibly on a set X and $A \subset X$. Then there exists a positive, finitely additive, G-invariant measure μ on X with $\mu(A) = 1$ if and only if A does not admit a paradoxical decomposition.* Here, a subset B of X admits a *paradoxical decomposition* (p.d.) if there exists a partition $A_1, \ldots, A_m, B_1, \ldots, B_n$ of B and elements $x_1, \ldots, x_m, y_1, \ldots, y_n$ of G such that both $\{x_i A_i: 1 \leqslant i \leqslant m\}$ and $\{y_i B_i: 1 \leqslant i \leqslant n\}$ are partitions of B. (Thus, by using G-translates, we can "pack" two copies of B into itself.) In the above circumstances it is convenient to say that A_i, B_i, x_i, y_i *is* a p.d. (for B with respect to G). An immediate consequence of Tarski's theorem is that *a (discrete) group G is not amenable if and only if G admits a p.d.* This beautiful result thus characterizes amenability directly in terms of translates of subsets of G with no mention of invariant means or measures. Tarski's proof uses a deep set-theoretic result of D. König [3]. Is there a simpler proof available?

A natural question, raised by W. R. Emerson, is the topological analogue of the above nonamenability theorem. Let G be a locally compact group. Let us say that G admits a *Borel* p.d. if there exists a p.d. as above with every A_i, B_i a *Borel* subset of G. The question then is: *Is it true that G is not amenable if and only if G admits a Borel p.d.*? The object of this note is to show that the answer to this question is yes.

What about a topological analogue for Tarski's theorem? The reader is referred to [2] for information about amenable locally compact groups.

THEOREM. *Let G be a locally compact group. Then G is not amenable if and only if G admits a Borel p.d.*

PROOF. Trivially, if G admits a Borel p.d., then G is not amenable. Conversely, suppose that G is not amenable. Since G is the (directed) union of its σ-compact, open subgroups, there exists a σ-compact, nonamenable, open subgroup H of G. Suppose that the result is true for H, and let A_i, B_i, x_i, y_i be a Borel p.d. for H as above. Let T be a transversal for the right H-cosets in G. One readily checks that $A_i T$, $B_i T$, x_i, y_i is a p.d. for G. To show that this p.d. is Borel, we need only show

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that AT is Borel in G if A is Borel in H. This is obvious if A is open in H, since then AT is open in G, and the result for general A follows by using the monotone class lemma. (Note that if $\{C_n\}$ is a decreasing sequence of subsets of H, then $\bigcap_{n=1}^{\infty}(C_nT) = (\bigcap_{n=1}^{\infty} C_n)T$.)

Thus G admits a Borel p.d., so we can suppose that $G = H$—i.e., G is σ-compact.

Since G is σ-compact, we can find a compact, normal subgroup K of G with G/K separable. Since K is amenable and G is not amenable, we have G/K not amenable. Let Q: $G \to G/K$ be the quotient map. If there exists a Borel p.d. involving sets A_i, B_i, then, by considering $Q^{-1}(A_i)$, $Q^{-1}(B_i)$, we see that G admits a Borel p.d. We can therefore suppose that G is separable.

Let G_e be the identity component of G. Then G/G_e is totally disconnected, and so contains a compact open subgroup L. Let Φ: $G \to G/G_e$ be the quotient map and $H = \Phi^{-1}(L)$. Then H is an almost connected, open and closed subgroup of G. There are two cases to be considered.

(i) *H is not amenable*. A result of Rickert [**5**, **6**] shows that there exists a discrete subgroup F of H isomorphic to the free group F_2 on two generators. In particular, F is closed in H. Now H is separable since G is, and a result of [**4**] yields a Borel cross section B for the right F-cosets in H. Now F is, of course, not amenable, and so by Tarski's theorem, we can find a p.d. A_i', B_i', x_i, y_i for F. Then $A_i'B$, $B_i'B$, x_i, y_i is a p.d. for H, and the p.d. is Borel since each A_i', B_i' is countable and B is Borel. We then produce a Borel p.d. for G as in the second paragraph of the present proof.

(ii) *H is amenable*. The group G acts on the discrete space G/H in the usual way. We claim that there does not exist a G-invariant mean on $\ell_\infty(G/H)$. (Indeed, following the usual line of argument in this context, if m were such a mean, and n was a left invariant mean on the space $C(H)$ of bounded, continuous, complex-valued functions on H, then the map $\phi \to m(xH \to n((\phi x)|_H))$, where $\phi x(y) = \phi(xy)$ $(x, y \in G)$, is a left invariant mean on $C(G)$, giving G amenable and, hence, a contradiction.) By Tarski's theorem we can find a p.d. A_i, B_i, x_i, y_i for G/H with respect to G. Then A_iH, B_iH, x_i, y_i is a Borel p.d. for G, and we are finished.

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