NONAMENABILITY AND BOREL PARADOXICAL
DECOMPOSITIONS FOR LOCALLY COMPACT GROUPS

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Abstract. We show that a locally compact group G is not amenable if and only if it
admits a Borel paradoxical decomposition.

In 1938 A. Tarski [7] proved the following remarkable theorem. Let G be a group
acting invertibly on a set X and A ⊂ X. Then there exists a positive, finitely additive,
G-invariant measure μ on X with μ(A) = 1 if and only if A does not admit a
paradoxical decomposition. Here, a subset B of X admits a paradoxical decomposi-
tion (p.d.) if there exists a partition A_1, ..., A_m, B_1, ..., B_n of B and elements
x_1, ..., x_m, y_1, ..., y_n of G such that both \{x_i A_i : 1 ≤ i ≤ m\} and \{y_i B_i : 1 ≤ i ≤ n\}
are partitions of B. (Thus, by using G-translates, we can “pack” two copies of B
into itself.) In the above circumstances it is convenient to say that A_i, B_i, x_i, y_i is a
p.d. (for B with respect to G). An immediate consequence of Tarski’s theorem is that
a (discrete) group G is not amenable if and only if G admits a p.d. This beautiful
result thus characterizes amenability directly in terms of translates of subsets of G
with no mention of invariant means or measures. Tarski’s proof uses a deep set-theo-
retic result of D. König [3]. Is there a simpler proof available?

A natural question, raised by W. R. Emerson, is the topological analogue of the
above non amenability theorem. Let G be a locally compact group. Let us say that G
admits a Borel p.d. if there exists a p.d. as above with every A_i, B_i a Borel subset of
G. The question then is: Is it true that G is not amenable if and only if G admits a
Borel p.d.? The object of this note is to show that the answer to this question is yes.

What about a topological analogue for Tarski’s theorem? The reader is referred to

Theorem. Let G be a locally compact group. Then G is not amenable if and only if G
admits a Borel p.d.

Proof. Trivially, if G admits a Borel p.d., then G is not amenable. Conversely,
suppose that G is not amenable. Since G is the (directed) union of its σ-compact,
open subgroups, there exists a σ-compact, nonamenable, open subgroup H of G.
Suppose that the result is true for H, and let A_i, B_i, x_i, y_i be a Borel p.d. for H as
above. Let T be a transversal for the right H-cosets in G. One readily checks that
A_i T, B_i T, x_i, y_i is a p.d. for G. To show that this p.d. is Borel, we need only show

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that $AT$ is Borel in $G$ if $A$ is Borel in $H$. This is obvious if $A$ is open in $H$, since
then $AT$ is open in $G$, and the result for general $A$ follows by using the monotone
class lemma. (Note that if $\{C_n\}$ is a decreasing sequence of subsets of $H$, then
$\cap_{n=1}^{\infty} (C_n T) = (\cap_{n=1}^{\infty} C_n) T$.)

Thus $G$ admits a Borel p.d., so we can suppose that $G = H$—i.e., $G$ is $\sigma$-compact.

Since $G$ is $\sigma$-compact, we can find a compact, normal subgroup $K$ of $G$ with $G/K$
separable. Since $K$ is amenable and $G$ is not amenable, we have $G/K$ not amenable.
Let $Q: G \to G/K$ be the quotient map. If there exists a Borel p.d. involving sets $A_i,$
$B_i$, then, by considering $Q^{-1}(A_i), Q^{-1}(B_i)$, we see that $G$ admits a Borel p.d. We can
therefore suppose that $G$ is separable.

Let $G_e$ be the identity component of $G$. Then $G/G_e$ is totally disconnected, and so
contains a compact open subgroup $L$. Let $\Phi: G \to G/G_e$ be the quotient map and
$H = \Phi^{-1}(L)$. Then $H$ is an almost connected, open and closed subgroup of $G$.
There are two cases to be considered.

(i) $H$ is not amenable. A result of Rickert [5, 6] shows that there exists a discrete
subgroup $F$ of $H$ isomorphic to the free group $F_2$ on two generators. In particular, $F$
is closed in $H$. Now $H$ is separable since $G$ is, and a result of [4] yields a Borel cross
section $B$ for the right $F$-cosets in $H$. Now $F$ is, of course, not amenable, and so by
Tarski’s theorem, we can find a p.d. $A'_i, B'_i, x_i, y_i$ for $F$. Then $A'_iB, B'_iB, x_i, y_i$ is a
p.d. for $H$, and the p.d. is Borel since each $A'_i, B'_i$ is countable and $B$ is Borel. We
then produce a Borel p.d. for $G$ as in the second paragraph of the present proof.

(ii) $H$ is amenable. The group $G$ acts on the discrete space $G/H$ in the usual way.
We claim that there does not exist a $G$-invariant mean on $\ell_\infty(G/H)$. (Indeed,
following the usual line of argument in this context, if $m$ were such a mean, and $n$
was a left invariant mean on the space $C(H)$ of bounded, continuous, complex-valued functions on $H$, then the map $\phi \to m(xH \to n((\phi x)|_H)), where $\phi x(y) = 
\phi(xy)$ ($x, y \in G$), is a left invariant mean on $C(G)$, giving $G$ amenable and, hence, a
contradiction.) By Tarski’s theorem we can find a p.d. $A_i, B_i, x_i, y_i$ for $G/H$ with
respect to $G$. Then $A_iH, B_iH, x_i, y_i$ is a Borel p.d. for $G$, and we are finished.

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