PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 96, Number 1, January 1986

INNER MODEL OPERATORS AND THE CONTINUUM HYPOTHESIS

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ABSTRACT. Assuming AD, every inner model operator satisfies the continuum hypothesis.

We work in ZF + DC, and usually assume the axiom of determinacy (AD). For an introduction to AD, see [**3** or **7**]. In this paper, *degree* always means Turing degree. A *cone* of degrees is a set of the form $\{\mathbf{d}: \mathbf{d}_0 \leqslant_T \mathbf{d}\}$. AD implies that the filter generated by the cones is a (countably complete) ultrafilter [**7**, 7D.15]. *Almost everywhere* (*a.e.*) will always refer to this measure. A *real* is a subset of ω.

DEFINITION. An *inner model operator* (*or* IMO) is a function $\mathscr{M}: x \mapsto (M_x, \leqslant_x)$ with domain $P(\omega)$ such that, for all reals x, M_x is a transitive (set) model of ZFC, $\leqslant_x$ is a well-ordering of $P(\omega) \cap M_x$, and the following three properties hold:

(1) For any $x, y \subset \omega$, if $x \equiv_T y$ then $M_x = M_y$.

(2) For any $x \subset \omega$, $x \in M_x$.

(3) For any $x \subset \omega$, $f^x \in M_x$, where f^x is the function with domain $\{y \subset \omega: y \equiv_T x\}$ and such that $f^x(y)$ is $\leqslant_y$.

This concept was introduced by Steel [**9**]. The above definition is a variant of his. This concept is meant to be considered under the hypothesis of AD, or some weak form of that axiom. Usually one identifies IMO's which agree almost everywhere. The canonical example of an IMO is the map $x \mapsto L(x)$. Any construction which relativizes to a real in a degree-invariant way gives an IMO. We give some other examples below. The definition is intended to be a formalization of the notion of a "natural" model; assuming AD, it excludes many "unnatural" models such as forcing models.

Examples of inner model operators. 1. $x \mapsto L(x)$. This is an abuse of the language; what we really mean is $x \mapsto (L_{\aleph_1^V}(x), \leqslant_{L(x)})$, where $\leqslant_{L(x)}$ is the canonical constructibility ordering. We will similarly abuse the language in all the following examples. All these examples have canonical well-orderings. In this paper we will only be concerned with reals and sets of reals in the model, so we might as well cut the universe off at a strongly inaccessible cardinal. (AD implies that $\aleph_1$ will be strongly inaccessible in any transitive model of ZFC.)

2. $x \mapsto$ the minimal model of ZFC containing x.

3. $x \mapsto L(x^\#)$.

Received by the editors October 17, 1984.

1980 *Mathematics Subject Classification*. Primary 03E60; Secondary 03D80, 03E15, 03E45, 04A30.

[1]Research partially supported by NSF Grant MCS82-11328.

4. More generally, for any jump operator J, $x \mapsto L(J(x))$ is an IMO. A *jump operator* is a degree-invariant, $\leqslant_T$-increasing function from reals to reals with enough uniformity to make (3) in the definition of IMO hold. (We give a more precise definition later.)

5. $x \mapsto L[\mu, x]$, where μ is a fixed normal measure on some cardinal.

6. There are generalizations of the $L[\mu]$ construction to stronger large cardinal properties, for example Mitchell's models for coherent sequences of measurables and for hypermeasurables [**5**, **6**]. These give rise to IMO's.

7. $x \mapsto K(x)$, where $K(x)$ is the core model [**2**] relativized to x.

8. Assuming projective determinacy, there exist analogs of L for the higher levels of the analytical hierarchy. For example, $P(\omega) \cap L$ is the largest countable Σ^1_2 subset of $P(\omega)$; analogously, for any n, there is a model L^{2n} such that $P(\omega) \cap L^{2n}$ is the largest countable Σ^1_{2n} set. There are other classes of analogs, too, since generalizing different properties of L gives different models. The following is a list of the analogs of L which have appeared in the literature:

$M^2, M^3, M^4, M^5, \ldots$ [**1**].

$L^2, L^4, L^6, L^8, \ldots$ [**1**].

$H_1, H_3, H_5, H_7, \ldots$ [**7**, 8G].

$L(Q_1), L(Q_3), L(Q_5), L(Q_7), \ldots$ [**4**].

$M^2 = L^2 = H_1 = L(Q_1) = L$. These models all have canonical well-orderings and all relativize. Hence each of these models corresponds to an IMO.

9. There are also analogs of L, similar to those of Example 8, for pointclasses beyond the projective hierarchy, such as the inductive sets or $(\Sigma^2_1)^{L[\mathbf{R}]}$. These also give IMO's.

10. $x \mapsto \mathrm{HOD}(x)$.

Next we consider an example of something that is *not* an IMO. For the moment, work in ZF + DC + $(\forall x \subset \omega)$ ($x^\#$ exists). Then for any $x \subset \omega$ there is a $y \subset \omega$ such that y is Cohen-generic over $L(x)$. But is there an IMO $\mathscr{M}$ such that $M_x = L(x, y)$ where y is Cohen-generic over $L(x)$? Assuming the axiom of choice the answer is obviously yes—for each degree $\mathbf{d}$, just choose a y and a well-ordering of $L(\mathbf{d}, y)$. (Without the axiom of choice, one can prove the existence of a function taking each x to a y which is generic over $L(x)$, for example, by taking the least y with respect to the canonical well-ordering of $L(x^\#)$. However, this function is not degree-invariant.) Without choice, it is conceivable that the answer is no.

Assuming AD, no such IMO exists. The following theorem of Steel [**9**] rules out this and many other forcing models: For any IMO $\mathscr{M}$, for a.e. x, either $(P(\omega) \cap M_x) \subset L(x)$ or $x^\# \in M_x$.

DEFINITION. $\mathscr{M} \models \psi$ means that, for a.e. x, $M_x \models \psi$.

Note that under AD, for any $\mathscr{M}$ and ψ, either $\mathscr{M} \models \psi$ or $\mathscr{M} \models \neg\psi$.

THEOREM. *Assume AD. Let $\mathscr{M}$ be an IMO. Then $\mathscr{M} \models CH$.*

Remarks and questions. 1. Woodin has shown (assuming ZF + DC + $(\forall x \subset \omega)$ ($x^\#$ exists)) that there is an IMO $\mathscr{N}$ such that $\mathscr{N} \models 2^{\aleph_2} > \aleph_3$. This question is open for $2^{\aleph_1} = \aleph_2$ and also open for $\diamondsuit$. Woodin's models N_x are obtained by forcing over $L(x)$; the forcing used has the property that the generic object can be built in a degree-invariant way.

2. Assuming $\mathrm{AD}_{\mathbf{R}}$, there exist degree-invariant maps $x \mapsto M_x$ where $M_x \models (\mathrm{ZF} + \neg\,\mathrm{AC})$, and in fact $M_x \models (\mathrm{ZF} + \mathrm{DC} + \mathrm{AD})$. This is proved in Solovay [**8**].

3. For examples (1)–(9) it was already known that CH holds in these models. Example (10) gives a new result. It was known that $\mathrm{HOD}^{L[\mathbf{R}]} \models \mathrm{CH}$, and similarly relativized to x, but it was not known for other HOD's, e.g. the HOD of the world of $\mathrm{AD}_{\mathbf{R}}$ (which is not equal to $\mathrm{HOD}^{L[\mathbf{R}]}$). The fact that $\mathrm{HOD}(x) \models \mathrm{CH}$, for a.e. x, leads to an obvious question: What about $x = 0$? Assuming $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}_{\mathbf{R}} + V = L[P(\mathbf{R})]$, does $\mathrm{HOD} \models \mathrm{CH}$?

4. Under any reasonable notion of definability, definable determinacy implies that *definable* IMO's satisfy CH. For example, projective determinacy implies that, for any projective IMO $\mathscr{M}$, $\mathscr{M} \models \mathrm{CH}$. $\mathscr{M}$ is *projective* if the following two relations on reals are projective:

$$\{(x, y)\colon y \in M_x\}, \qquad \{(x, y, z)\colon y \in M_x \,\&\, z \in M_x \,\&\, y \leqslant_x z\}.$$

Note that examples (1)–(3) and (5)–(8) are projective IMO's.

The rest of this paper consists of a proof of the theorem. We first need a sequence of lemmas. We work in ZF + DC, and state additional axioms in the hypotheses of the lemmas.

LEMMA 1. *Assume the axiom of choice. Let $C \subset P(\omega)$. Suppose that for any x, $y \in C$, either $x \leqslant_T y$ or $y \leqslant_T x$. Then* $\mathrm{card}(C) \leqslant \aleph_1$.

DEFINITION. A *jump operator* is a pair $(\mathbf{d}, J)$ where $\mathbf{d}$ is a degree and J is a function from $\{x \subset \omega\colon \mathbf{d} \leqslant_T x\}$ into $P(\omega)$ such that:

(1) J is uniformly degree-invariant, i.e. there is a function $f\colon \omega \to \omega$ such that, for all $e \in \omega$, for all x, $y \geqslant_T \mathbf{d}$, if $x \equiv_T y$ via e then $J(x) \equiv_T J(y)$ via $f(e)$,

(2) J is $\leqslant_T$-increasing, i.e., for all $x \geqslant_T \mathbf{d}$, $x \leqslant_T J(x)$.

LEMMA 2 (STEEL [**9**]). *Assume AD. If J_1 and J_2 are jump operators, then for a.e. x, either $J_1(x) \leqslant_T J_2(x)$ or $J_2(x) \leqslant_T J_1(x)$.*

LEMMA 3. *Let $\mathscr{M}$ be an IMO. There are functions $x \mapsto C_x$, $x \mapsto \xi_x$, and $x \mapsto \phi_x$ with the following properties*:

(1) *For all $x \subset \omega$,*

(a) *$C_x \subset (P(\omega) \cap M_x)$, ξ_x is an ordinal, and ϕ_x is a one-to-one function from C_x onto ξ_x,*

(b) *C_x, ξ_x, and ϕ_x are in M_x,*

(c) $M_x \models \mathrm{card}(C_x) = 2^{\aleph_0}$,

(d) *for all $z \in C_x$, $x \leqslant_T z$.*

(2) *For all x, $y \subset \omega$ such that $x \equiv_T y$,*

(a) $\xi_x = \xi_y$,

(b) *for any v and w, if $v \in C_x$, $w \in C_y$, and $\phi_x(v) = \phi_y(w)$ then $v \equiv_T w$. Moreover, this is uniform in the following sense. There is a function $g\colon \omega \to \omega$ such* *that for any v and w and any $e \in \omega$, if $v \in C_x$, $w \in C_y$, $\phi_x(v) = \phi_y(w)$ and $x \equiv_T y$ via e, then $v \equiv_T w$ via $g(e)$.*

PROOF. Let $\mathscr{M}$: $x \mapsto (M_x, \leqslant_x)$ be an IMO. Let

$$\xi_x = \sup\{\text{order type of } \leqslant_y : y \equiv_T x\}.$$

For $\eta < \xi_x$, let v_x^η be the real which is the join of x'' and

$$\{(e, i) \in \omega^2 : (\exists y, w \subset P(\omega))(y \equiv_T x \text{ via } e \,\&\, w \in M_x \\ \&\text{ the rank of } w \text{ in } \leqslant_y \text{ is } \eta \,\&\, i \in w)\}.$$

Then let $C_x = \{v_x^\eta : \eta < \xi_x\}$ and let $\phi_x(v_x^\eta) = \eta$. This works. It follows from (2) and (3) in the definition of IMO that everything is in the model M_x. The reason (1c) holds is that every real in M_x is recursive in an element of C_x.

PROOF OF THEOREM. Fix an IMO $\mathscr{M}$ and fix maps $x \mapsto C_x$, $x \mapsto \xi_x$, and $x \mapsto \phi_x$ satisfying Lemma 3 for $\mathscr{M}$. By applying Lemma 1 inside the model M_x, we see that to show that $M_x \vDash \mathrm{CH}$, it will suffice to show that

$$(*) \qquad (\forall z_1, z_2 \in C_x)(\text{either } z_1 \leqslant_T z_2 \text{ or } z_2 \leqslant_T z_1).$$

Note that by part (2) of Lemma 3, $(*)$ depends only on the Turing degree of x. Suppose that the theorem is false. Then $(*)$ is false on a cone, say the cone above $\mathbf{d}_0$. For every $x \geqslant_T \mathbf{d}_0$, let $z_1(x)$ and $z_2(x)$ be the ϕ_x-least counterexample to $(*)$. By Lemma 3(1d), the maps $x \mapsto z_i(x)$ are $\leqslant_T$-increasing, and by Lemma 3(2b) they are uniformly degree-invariant, that is, they are jump operators. For $x \geqslant_T \mathbf{d}_0$, $z_1(x) \nleqslant_T z_2(x)$ and $z_2(x) \nleqslant_T z_1(x)$. This contradicts Lemma 2.

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