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PROJECTIVE MODULES WITH FREE MULTIPLES AND POWERS

HYMAN BASS AND ROBERT GURALNICK

ABSTRACT. Let P be a finitely generated projective module over a commutative ring. Some tensor power of P is free iff some sum of copies of P is free.

THEOREM. *Let A be a commutative ring and P a finitely generated projective A-module of rank $r > 0$.*

(a) *Suppose that $P^{\otimes n}$ ($= P \otimes_A \cdots \otimes_A P$, n times) is free for some integer $n > 0$. Then mP ($= P \oplus \cdots \oplus P$, m times) is free for some integer $h > 0$, where $m = r^{h(n-1)} \cdot n^h$.*

(b) *Suppose that mP is free for some integer $m > 0$. Then for some integer $t > 0$, $P^{\otimes m^t}$ is free.*

Since P and the assumed isomorphisms with free modules are defined over some finitely generated subring of A, there is no loss in assuming, as we do, that A is noetherian, say of dimension d.

In $K_0(A)$ we put $x = [P]$ and $y = x - r$.

PROOF OF (a). By hypothesis, $x^n = r^n$ so

$$0 = x^n - r^n = yz, \tag{1}$$

where $z = x^{n-1} + x^{n-2}r + \cdots + xr^{n-2} + r^{n-1}$. We have

$$\begin{aligned} z - nr^{n-1} &= \sum_{i=1}^{n} (x^{n-i}r^{i-1} - r^{n-1}) \\ &= \sum_{i=1}^{n} r^{i-1}(x^{n-i} - r^{n-i}) = -yw \end{aligned}$$

for some w, whence $n \cdot r^{n-1} = z + yw$. For $h > 0$ we thus have

$$n^h r^{h(n-1)} = zu + y^h w^h \tag{2}$$

for some u. According to [**B**, Chapter IX, Proposition (4.4)], y is nilpotent, in fact $y^{h+1} = 0$ for some $h \leqslant d = \dim(A)$. It follows then from (1) and (2) that, with $m = n^h r^{h(n-1)}$, we have $my = 0$, i.e. $mx = mr$, i.e. $[mP] = [mA^r]$. Enlarging h, if necessary, so that $mr > d$, it then follows from the Cancellation Theorem [**B**, Chapter III, Corollary (3.5)], that $mP \cong mA^r = A^{mr}$.

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Proof of (b). By assumption, $mx = mr$, i.e. $my = 0$. By induction on $t \geqslant 1$ we can write $x^{m^t} = r^{m^t} + y^{2^t} z_t$ for some $z_t \in K_0(A)$. Indeed $x = r + y$ so $z_1 = 1$. By induction,

$$\begin{aligned} x^{m^{t+1}} &= \left(x^{m^t}\right)^m = \left(r^{m^t} + y^{2^t} z_t\right)^m \\ &= \left(r^{m^t}\right)^m + my^{2^t} z_t \left(r^{m^t}\right)^{m-1} + \left(y^{2^t}\right)^2 z_{t+1}, \end{aligned}$$

whence the claim, since $my = 0$. Now as above, $y^h = 0$ for some $h \leqslant d + 1$. Taking t large enough so that $2^t \geqslant h$, we then have $x^{m^t} = r^{m^t}$, i.e. $[P^{\otimes m^t}] = [(A^r)^{\otimes m^t}] = [A^{r^{m^t}}]$. If $r = 1$ then $P^{\otimes m^t} \cong A$, so assume that $r > 1$. With t large enough so that $r^{m^t} > d = \dim(A)$ we again conclude from the Cancellation Theorem that $P^{\otimes m^t}$ is free.

References

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Department of Mathematics, Columbia University, New York, New York 10027

Department of Mathematics, University of Southern California, Los Angeles, California 90089