NORMAL SUBGROUPS OF THE GENERAL LINEAR GROUPS
OVER VON NEUMANN REGULAR RINGS

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ABSTRACT. Let $A$ be a von Neumann regular ring or, more generally, let $A$
be an associative ring with 1 whose reduction modulo its Jacobson radical is
von Neumann regular. We obtain a complete description of all subgroups of
$GL_n A$, $n \geq 3$, which are normalized by elementary matrices.

1. Introduction. For any associative ring $A$ with 1 and any natural number $n$,
let $GL_n A$ be the group of invertible $n$ by $n$ matrices over $A$ and $E_n A$
the subgroup generated by all elementary matrices $x^{i,j}$, where $1 \leq i \neq j \leq n$
and $x \in A$.

In this paper we describe all subgroups of $GL_n A$ normalized by $E_n A$
for any von Neumann regular $A$, provided $n \geq 3$. Our description is standard (see Bass [1] and
Vaserstein [14, 16]): a subgroup $H$ of $GL_n A$ is normalized by $E_n A$ if and only if
$H$ is of level $B$ for an ideal $B$ of $A$, i.e. $E_n(A, B) \subset H \subset G_n(A, B)$. Here $G_n(A, B)$
is the inverse image of the center of $GL_n(A/B)$ (when $n \geq 2$, this center consists
of scalar invertible matrices over the center of the ring $A/B$) under the canonical
homomorphism $GL_n A \to GL_n(A/B)$ and $E_n(A, B)$ is the normal subgroup of $E_n A$
generated by all elementary matrices in $G_n(A, B)$ (when $n \geq 3$, the group $E_n(A, B)$
is generated by matrices of the form $(-y)^{i,j} x^{i,j} y^{j,i}$ with $x \in B, y \in A, 1 \leq i \neq j \leq n$,
see [14]).

Recall that a ring $A$ is called von Neumann regular (see von Neumann [13],
Goodearl [7]) if for any $z$ in $A$ there is $x$ in $A$ such that $zxz = z$. Then every factor
ring and every ideal of $A$ is also von Neumann regular.

In fact, to be more general, we assume that $A/\text{rad}(A)$ (rather than $A$) is von
Neumann regular, where rad means the Jacobson radical. For example, this as-
sumption holds for any Artinian ring $A$ or for any commutative semilocal ring $A$.

THEOREM 1. Assume that $A/\text{rad}(A)$ is von Neumann regular and $n \geq 2$. Then
for any ideal $B$ of $A$:

(a) $E_n(A, B)$ contains all matrices of the form $1_n + vu$, where $v$ is an $n$-column
over $A$, $u$ is an $n$-row over $B$, and $uv = 0$; in particular, $E_n(A, B)$ is normal in
$GL_n A$;

(b) $E_n(A, B) \supset [E_n A, G_n(A, B)]$; in particular, every subgroup of $GL_n A$
of level $B$ is normalized by $E_n A$;

(c) if $n \geq 3$, we have $E_n(A, B) = [E_n A, E_n B] = [GL_n A, E_n(A, B)] = [E_n A, H]$
for any subgroup $H$ of level $B$, where $E_n B$ is the subgroup of $G_n(A, B)$ generated
by elementary matrices;

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(d) if $A$ is von Neumann regular, we have $E_n B = E_n(A, B)$; if moreover, $n \geq 3$, we have $E_n B = [E_n B, E_n B]$.

**Theorem 2.** Assume that $A/\text{rad}(A)$ is von Neumann regular and $n \geq 3$. Then every subgroup $H$ of $\text{GL}_n A$ normalized by $E_n A$ is of level $B$ for some ideal $B$ of $A$, i.e. $E_n(A, B) \subset H \subset G_n(A, B)$.

Note that a subgroup $H$ of $\text{GL}_n A$, $n \geq 2$, cannot be of level $B$ and of level $B'$ for two distinct ideals $B$ and $B'$ of $A$. So the level $B$ in Theorem 2 is unique.

Theorems 1 and 2 were proved by Dickson [2] when $A$ is a field (the condition $n \geq 3$ in this case can be replaced by the condition $\text{card}(A) \geq 4$), by Dieudonné [3] when $A$ is a division ring, by Klingenberg [10] when $A$ is a commutative local ring, by Bass [1] when $A$ satisfies the stable range condition $sr(A) \leq n - 1$, by Vaserstein [14] when central localizations of $A$ satisfy this stable range condition (for example, when $A$ is finite as module over its center) and $n \geq 3$, and by Vaserstein [16] when $A$ is a Banach algebra. Theorem 2 is claimed by Golubchik [5, 6] under the additional condition that $A/M$ is an Ore ring for every maximal ideal $M$ of $A$.

Note that von Neumann regular rings $A$ satisfying $sr(A) \leq 1$ are known as unit regular rings, see [7, 8, 9, 11, 12, 15].

**2. Proof of Theorem 1(a).** We write

$$v = (v_i) = \left( \begin{array}{c} v' \\ v_n \end{array} \right) \quad \text{and} \quad u = (u_j) = (u', u_n)$$

with $v_i$ in $A$ and $u_j$ in $B$.

**Case 1.** $1 + v_n u_n \in \text{GL}_1 B$. We set \( d := 1 + v_n u_n \), \( d' := 1 + u_n v_n = 1 - v' u' \in \text{GL}_1 B \) (see [17, §2]) and $a = 1_{n-1} + v' u' - v' u_n d^{-1} v_n u' = 1_{n-1} + v' (1 - u_n d^{-1} v_n) u' = 1_{n-1} + v' d'^{-1} u'$. Then

$$1_n + vu = \left( \begin{array}{c} 1_{n-1} + v' u' \\ v' u_n \end{array} \right)$$

$$= \left( \begin{array}{c} 1_{n-1} \\ 0 \end{array} \right) \left( \begin{array}{cc} a & 0 \\ 0 & d \end{array} \right) \left( \begin{array}{c} 1_{n-1} \\ d^{-1} v_n u' \\ 1 \end{array} \right)$$

$$\in E_n B \left( \begin{array}{c} a \\ 0 \\ d \end{array} \right) E_n B.$$

We have to prove that \( \left( \begin{array}{c} a \\ 0 \\ d \end{array} \right) \in E_n(A, B) \).

Since $1 + u' v' d'^{-1} = d'^{-1}$, we have

$$\left( \begin{array}{cc} a & 0 \\ 0 & 1 \end{array} \right) = \left( \begin{array}{c} 1_{n-1} \\ u' \end{array} \right) \left( \begin{array}{cc} 1_{n-1} & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 1_{n-1} & 0 \\ 0 & 1 \end{array} \right)$$

$$\left( \begin{array}{cc} a & 0 \\ 0 & 1 \end{array} \right) = \left( \begin{array}{c} 1_{n-1} \\ v' \end{array} \right) \left( \begin{array}{cc} 1_{n-1} & 0 \\ 0 & d'^{-1} \end{array} \right)$$

$$\in E_n(A, B) \left( \begin{array}{c} 1_{n-1} \\ 0 \end{array} \right)$$.
By [17, §2],
\[
\begin{pmatrix}
1_{n-1} & 0 \\
0 & d^{-1}d
\end{pmatrix} \in E_n(A, B).
\]
So \(1_n + vu \in E_n(A, B)\) in Case 1.

**Case 2.** \(v_i \in \text{rad}(A)\) for some \(i\) with \(1 \leq i \leq n\). Since \(E_n(A, B)\) is normalized by all permutation matrices, we can assume that \(i = n\). Then \(1 + vnun \in GL_1 B\), so we are reduced to Case 1.

**General case.** We now use the condition that \(A/\text{rad}(A)\) is von Neumann regular, hence there is an \(x \in A\) such that \(vnxvn - vn \in \text{rad}(A)\). Then \(1 + vn(1 - xvn)un \in GL_1 B\), hence \(g := 1_n + v(1 - xvn)u \in GL_n B\) by Case 1.

Also we have
\[
(-vn_{-1}x)^n_{-1,n}(1_n + vxvnu)(vn_{-1}x)^n_{-1,n} = 1_n + ((-vn_{-1}x)^n_{-1,n}vu(vn_{-1}x)^n_{-1,n})(u(vn_{-1}x)^n_{-1,n})
\]
and
\[
((-vn_{-1}x)^n_{-1,n}vu,xvn_{-1})n_{-1,n} = vn_{-1}(1 - xvn)xvn = vn_{-1}x(vn - vnvn) \in \text{rad}(A),
\]
hence \(h := 1_n + vxvnw \in E_n(A, B)\) by Case 2 with \(i = n - 1\).

Therefore \(1_n + vu = gh \in E_n(A, B)\).

**3. Proof of Theorem 1(b).** It suffices to show that \([y^{i,j}, g] := y^{i,j}g(-y^{i,j}g^{-1}) \in E_n(A, B)\) for any elementary \(y^{i,j}\) in \(E_n A\) and any \(g \in G_n(A, B)\). Since \(E_n(A, B)\) is normalized by all permutation matrices, we can assume that \((i, j) = (1, n)\).

Then \([y^{i,j}, g] = y^{i,n}(1_n - vvw)\), where \(v = (v', v_n)\) is the first column of \(g\) and \(w = (w', w_n)\) is the last row of \(g^{-1}\), so \(vw = 0\).

As in the end of the previous section, we find \(x \in A\) such that \(vnxvn - vn \in \text{rad}(A)\), and we have \(h := 1_n - vxvnw \in E_n(A, B)\), hence
\[
[(xvn)^{1,n}, g] = (xvn)^{1,n}(1_n - vxvnw) \in E_n(A, B),
\]
i.e. \((xvn)^{1,n}\) and \(g\) commute modulo \(E_n(A, B)\).

To complete our proof, it suffices to show that \((1 - xvn)^{1,n}\) also commutes with \(g\) modulo \(E_n(A, B)\). We set \(u := -(1 - xvn)w = (u', u_n)\). Then
\[
[(1 - xvn)^{1,n}, g] = (1 - xvn)^{1,n}(1_n + vu),
\]
with \(vnun = vn(1 - xvn)w \in \text{rad}(B)\), hence \(d := 1 + vnun \in GL_1 B\). Also \(v_i \in B\) for \(i \geq 2\), \(u_j \in B\) for \(j \leq n - 1\) and \(v_i u_n + 1 \in B\).

We set \(d' := 1 + u_nvn = 1 - u'v' \in GL_1 B\) and \(a := 1_{n-1} + v'u' - v'u'd^{-1}v_nu' = 1_{n-1} + v'd^{-1}u'\). Then
\[
(1 - xvn)^{1,n}(1_n + vu) = (1 - xvn)^{1,n}\begin{pmatrix}
1_{n-1} & v'u'd^{-1} \\
0 & 1
\end{pmatrix}\begin{pmatrix}
a & 0 \\
0 & d
\end{pmatrix}\begin{pmatrix}
1_{n-1} & 0 \\
0 & d^{-1}v_nu' \\
1
\end{pmatrix} \in E_n B.
\]

Now, as in the previous section (see Case 1 there), we see that \((a, 0, 0) \in E_n(A, B)\).
(Note that \(u'\) is an \((n - 1)\)-row over \(B\).)
4. **Proof of Theorem 1(c).** In the view of Theorem 1(a), (b), we have only the inclusion \( E_n B \subset [E_n A, E_n B] \) to prove. But we have it for any ring \( A \) with 1 and any \( n \geq 3 \) by the formula \( x^{i,j} = [1^{i,k}, x^{k,j}] \), where \( 1 \leq i \neq j \neq k \neq i \leq n \) and \( x \in B \).

5. **Proof of Theorem 1(d).** We want to prove first that \( E_n (A, B) = E_n B \), i.e. \( E_n B \) is normalized by every elementary matrix \( y^{i,j} \) in \( \text{GL}_n A \). Since \( E_n B \) is normalized by all permutation matrices, we can assume that \((i,j) = (1,2)\). It suffices to prove that \( h := (-y)^{1,2} y^{1,2} \in E_n B \) for every elementary matrix \( g \) in \( E_n B \). This is trivial (and true for an arbitrary ring \( A \)) unless \( g = z^{2,1} \) where \( z \in B \). In this case we can assume that \( n = 2 \).

Since \( A \) is von Neumann regular, \( z = zz z \) for some \( x \) in \( A \). We have

\[
 h = (-y)^{1,2} z^{2,1} y^{1,2} = (xyz - y)^{1,2}(y - xzy)^{1,2}(xzy)^{1,2}.
\]

But \( (xyz)^{1,2} \in E_2 B \) and

\[
 (xyz - y)^{1,2} z^{2,1} (y - xzy)^{1,2} = \begin{pmatrix} 1 + (xz - 1)yz & 0 \\ z & 1 \end{pmatrix} = ((xz - 1)yzx)^{1,2} z^{2,1}(1 - xz)yz^{1,2} \in E_2 B.
\]

When \( n \geq 3 \), for any elementary \( z^{i,j} \) in \( E_n B \) we have \( z^{i,j} = [(zx)^{i,k}, z^{k,j}] \), where \( k \neq i, j \) and \( z = zz z \) with \( x \) in \( A \).

6. **Proof of Theorem 2.** Let \( H \) be a subgroup of \( \text{GL}_n A \) normalized by \( E_n A \), where \( n \geq 3 \). The condition that \( A/\text{rad}(A) \) is von Neumann regular will not be used in Cases 1–5 of Lemma 3 below or Lemma 4.

**Lemma 3.** If \( H \) is not central, then \( H \) contains an elementary matrix \( \not \in 1_n \).

**Proof.** Case 1. \( H \ni g = (g_{i,j}) \) such that \( g_{n,1} = 0 \) and \( g \) does not commute with some \( 1^{k,1} \in E_n A \). Then \( H \) contains an elementary matrix \( \not \in 1_n \) by Vaserstein [14].

Case 2. \( H \ni h = (h_{i,j}) \) such that \( h_{n,2} \not \in 0 \) and \( h_{n,1} + h_{n,2} y = 0 \) for some \( y \) in \( A \). Then \( H \ni (-y)^{2,1} h z^{2,1} = g = (g_{i,j}) \) and \( g_{n,1} = h_{n,1} + h_{n,2} y = 0, g_{n,2} = h_{n,2} \not \in 0 \), so \( [g, 1^{2,1}] \neq 1_n \). Thus, we are reduced to Case 1.

Case 3. \( H \) contains a noncentral \( g = (g_{i,j}) \) with \( g_{n,1} = 0 \). If \( g \) does not commute with some \( 1^{k,1} \in E_n A \), we are done by Case 1. Otherwise, \( g \) is a scalar matrix: \( g_{i,j} = 0 = g_{i,j} - g_{j,i} \) for all \( i \neq j \). Since \( g \) does not belong to the center of \( \text{GL}_n A \), there is \( y \) in \( A \) such that \( yg_{i,1} \neq g_{1,i} y \). Then \( [g, y^{1,2}] = (g_{1,1} y - yg_{2,2})^{1,2} \not \in 1_n \) is an elementary matrix in \( H \).

Case 4. \( H \) contains a noncentral \( h = (h_{i,j}) \) with \( h_{2,2} \in \text{GL}_1 A \). If \( (h^{-1})_{n,1} = 0 \), we are done by Case 3 with \( g = h^{-1} \). Otherwise, \( H \ni (-1)^{1,2} h^{1,2} = (g_{i,j}) \) with \( (g_{n,1}, g_{n,2}) = (h^{-1})_{n,1} (h_{2,1}, h_{2,2}) \), so we are reduced to Case 2.

Case 5. \( H \) contains a noncentral \( h = (h_{i,j}) \) with \( h_{n,2} = 0 \). Since \( f := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \in E_2 A \), we have \( f' := \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix} \in E_n A \) and \( g := f' h f'^{-1} \in H \).

Since \( g_{n,1} = h_{n,2} = 0 \), we are reduced to Case 3.

General case. We pick a noncentral \( h = (h_{i,j}) \) in \( H \) and find \( x \) in \( A \) such that \( z := h_{n,2} x h_{n,2} - h_{n,2} \in \text{rad}(A) \). We set \( p := 1 - h_{n,2} x \). If \( ph_{n,1} = 0 \), i.e. \( h_{n,1} - h_{n,2} x h_{n,1} = 0 \), then we are done by Case 5 or Case 2. Otherwise, the matrix
\[ g = (g_{i,j}) := h^{-1}p^{1,n}h(-p)^{1,n} \in H \text{ is not central and } g_{2,2} = 1 + (h^{-1})_{2,1}ph_{n,2} = 1 - (h^{-1})_{2,1}e \in \text{GL}_1 A, \text{ so we are reduced to Case 4.} \]

**Lemma 4.** If \( H \ni x^i,j, \text{ where } x \in A, 1 \leq i \neq j \leq n, \text{ then } H \supset E_n(A,B), \text{ where } B \text{ is the (two-sided) ideal of } A \text{ generated by } x. \)

**Proof.** It follows easily from the identities \( y^{i,j}z^{i,j} = (y + z)^{i,j} \text{ and } [y^{i,j}, z^{i,k}] = (yz)^{i,k}, \) where \( 1 \leq i \leq j \neq k \neq i \leq n \) and \( y, z \in A \) (we use here that \( n \geq 3; \) no conditions on \( A \) are needed).

Now we can conclude our proof of Theorem 2. By Lemma 4, there is an ideal \( B \) of \( A \) such that \( E_n(A,B) \) contains all elementary matrices in \( H. \) Consider the image \( H' \) of \( H \) in \( \text{GL}_n(A/B). \) Since the ring \( (A/B)/\text{rad}(A) \) is a factor ring of \( A/\text{rad}(A), \) it is also von Neumann regular. Since \( H' \) is normalized by \( E_n(A/B) \) which is the image of \( E_nA, \) Lemma 3 applied to \( H' \) gives that either \( H' \) is central or \( H' \) contains an elementary matrix \( (x')^{i,j}, \) where \( 0 \neq x' \in A/B \) and \( 1 \leq i \neq j \leq n. \) In the latter case, \( H \ni x^{i,j}g, \) where \( 0 \neq x \in A, x' = x + B, \) and \( g \in \text{GL}_n(B). \) We pick an integer \( k \neq i, j \) in the interval \( 1 \leq k \leq n. \) Then \( H \ni [x^{i,j}g, 1^{i,k}] = x^{i,k}1^{i,j}x^{j,i}((-1)^{j,k}g(-x)_{i,j}(-1)^{j,k}E_n(A,B) \subset x^{i,k}H \text{ by Theorem 1(b).} \) Therefore \( H \ni x^{i,k} \) which contradicts our choice of \( B. \)

Thus, \( H' \) is central in \( \text{GL}_n(A/B), \) i.e. \( H \subset G_n(A,B). \)

**Remark.** From the proof of Theorem 1(a) (see §2 above), it is clear that the group \( E_n(A,B) \) is generated by matrices of the form \((-y)^{j,i}x^{j,i}y^{j,i} \) with \( x \in B \) and \( y \in A, \) provided \( n \geq 2 \) and \( A/\text{rad}(A) \) is von Neumann regular. If \( n \geq 3, \) no restrictions on \( A \) are needed.

**References**


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