MIXED HADAMARD’S THEOREMS
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Dedicated to Professor Hisaharu Umegaki on his sixtieth birthday and
in celebration of his having been honoured as an emeritus
Professor of Tokyo Institute of Technology

ABSTRACT. An operator $T$ means a bounded linear operator on a complex
Hilbert space $H$. We give two types of mixed Hadamard’s theorems containing
the terms $T$, $|T|$ and $|T^*|$ as extensions of Hadamard’s theorem and mixed
Schwarz’s inequality $|\langle Tx, y \rangle|^2 \leq |\langle T|x, x \rangle| |\langle T^*|y, y \rangle|$ for any $T$ and for any $x$
and $y$ in $H$. Also we scrutinize the cases when the equalities in these mixed
Hadamard’s theorems hold.

1. Statement of the results.

THEOREM 1 (MIXED HADAMARD’S TYPE 1). For any operator $T$ on $H$ and
any $x_1, x_2, \ldots, x_n$ in $H$, let $G_n$ be defined by

$$
G_n = \begin{vmatrix}
|T|x_1, x_1 & \langle T^*|x_2, x_2 & \langle T^*|x_3, x_3 & \cdots & \langle T^*|x_n, x_n \\
\langle T^*|x_2, x_1 & |T^*|x_2, x_2 & \langle T^*|x_3, x_3 & \cdots & \langle T^*|x_n, x_2 \\
\langle T^*|x_3, x_1 & \langle T^*|x_3, x_2 & |T^*|x_3, x_3 & \cdots & \langle T^*|x_n, x_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\langle T^*|x_n, x_1 & \langle T^*|x_n, x_2 & \langle T^*|x_n, x_3 & \cdots & |T^*|x_n, x_n \\
\end{vmatrix}
$$

Then

$$
0 \leq G_n \leq (|T|x_1, x_1) \prod_{j=2}^{n} (|T^*|x_j, x_j)
$$

and $G_n = 0$ if and only if $S_1 = \{ |T|x_1, T^*x_2, T^*x_3, \ldots, T^*x_n \}$ is a system of linearly
dependent vectors if and only if $S_2 = \{ T^*x_1, |T^*|x_2, |T^*|x_3, \ldots, |T^*|x_n \}$ is a system
of linearly dependent vectors. On the right-hand side, equality holds if and only if
$\langle Tx_j, x_j \rangle = 0$ for $j = 2, 3, \ldots, n$ and $|\langle T^*|x_j, x_k \rangle| = 0$ for $j < k$ ($j = 2, 3, \ldots, n-1$)
or $S_1$ contains the zero vector (equivalently, $S_2$ contains the zero vector).

THEOREM 2 (MIXED HADAMARD’S TYPE 2). For any operator $T$ on $H$ and
any $x_1, x_2, \ldots, x_n$ in $H$, let $G_{2n}$ be defined by

$$
G_{2n} = \begin{vmatrix}
|T|x_1, x_1 & \langle T|z_2, x_2 & \langle T|x_3, x_3 & \cdots & \langle T|z_{2n-1}, x_{2n-1} & \langle T|z_{2n}, x_{2n} \\
\langle T^*|z_2, x_1 & |T^*|z_2, x_2 & \langle T^*|z_3, x_3 & \cdots & \langle T^*|z_{2n-1}, x_{2n-1} & \langle T^*|z_{2n}, x_{2n} \\
\langle T|z_3, x_1 & \langle T|z_3, x_2 & |T|z_3, x_3 & \cdots & \langle T|z_{2n-1}, x_{2n-1} & \langle T|z_{2n}, x_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\langle T|z_{2n-1}, x_1 & \langle T|z_{2n-1}, x_2 & \langle T|z_{2n-1}, x_3 & \cdots & |T|z_{2n-1}, x_{2n} \\
\langle T^*|z_{2n-1}, x_2 & \langle T^*|z_{2n-1}, x_3 & \cdots & \langle T^*|z_{2n-1}, x_{2n} & |T^*|z_{2n}, x_{2n} \\
\end{vmatrix}
$$

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217
Then
\[ 0 \leq G_{2n} \leq \prod_{j=1}^{2n-1} (|T|x_j, x_j)(|T^*|x_{j+1}, x_{j+1}) \]
and \( G_{2n} = 0 \) if and only if \( S_1 = \{ |T|x_1, T^*x_2, |T|x_3, T^*x_4, \ldots, |T|x_{2n-1}, T^*x_{2n} \} \) is a system of linearly dependent vectors if and only if
\[ S_2 = \{ Tx_1, T^*x_2, T^*x_3, \ldots, Tx_{2n-1}, T^*x_{2n} \} \]
is a system of linearly dependent vectors. On the right-hand side, equality holds if and only if \( (|T|^jx_{2j}, x_{2k}) = 0 \) for \( j \neq k \), \( (|T|^jx_{2j-1}, x_{2k-1}) = 0 \) for \( j \neq k \), and \( (Tx_{2j-1}, x_{2k}) = 0 \) for \( j, k = 1, 2, \ldots, n \), or \( S_1 \) contains the zero vector (equivalently, \( S_2 \) contains the zero vector).

**Corollary 1 (Mixed Schwarz’s Inequality).** For any operator \( T \) and any \( x, y \) in \( H \), then
\[ \langle Tx, y \rangle^2 \leq \langle T|x, x \rangle \langle T^*|y, y \rangle. \]
The equality holds if and only if \( |T|x \) and \( T^*y \) are linearly dependent if and only if \( Tx \) and \( |T^*|y \) are linearly dependent.

**Remark.** We would like to emphasize that the equality holds if and only if \( |T|x \) and \( T^*y \) are linearly dependent if and only if \( Tx \) and \( |T^*|y \) are linearly dependent. One might believe that the equality would hold if and only if \( |T|x \) and \( |T^*|y \) are linearly dependent. But here we can give a simple counterexample as follows. Let
\[ T = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \quad x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 \\ 4 \end{pmatrix}. \]
Then
\[ |T^*|y = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2|T|x, \]
that is, \( |T|x \) and \( |T^*|y \) are linearly dependent, but
\[ \langle Tx, y \rangle^2 = 36 \neq \langle T|x, x \rangle \langle T^*|y, y \rangle = 54. \]
This mixed Schwarz’s inequality is discussed in [3, Problem 138] except the case when the equality holds.

**2. Proofs of the results.**
In order to show the results, we need the following

**Theorem A.** For \( x_1, x_2, \ldots, x_n \) in \( H \), let \( G_n \) be the determinant of a square matrix of order \( n \) defined by \( G_n = \|((x_j, x_k))\| \). Then
\[ 0 \leq G_n \leq \|x_1\|^2 \|x_2\|^2 \cdots \|x_n\|^2. \]
On the left-hand side, equality holds if and only if \( x_1, x_2, \ldots, x_n \) are linearly dependent. On the right-hand side, equality holds if and only if \( x_1, x_2, \ldots, x_n \) are mutually orthogonal or \( \{x_1, x_2, \ldots, x_n\} \) contains the zero vector.

The right-hand-side inequality in Theorem A is Hadamard’s theorem and also the left-hand-side inequality in Theorem A is well known and can be considered as a generalization of Schwarz’s inequality. Many ingenious and elegant proofs of Hadamard’s theorem have been given by many authors (for example [1, 2, 4, 5]).
**THEOREM B.** Let $T = U|T|$ be the polar decomposition of $T$ where $U$ means the partial isometry and $|T| = (T^*T)^{1/2}$ with $N(U) = N(|T|)$ where $N(S)$ denotes the kernel of an operator $S$. Then

(i) $|T^*| = U|T|U^* = UT^*$,

(ii) $T^* = U^*|T^*|$ is also the polar decomposition of $T^*$ with $N(U^*) = N(|T^*|)$.

**PROOF OF THEOREM B.** Theorem B is well known, but for the sake of convenience, we cite the proof. (i) As $U^*U$ is the initial projection and $U^*|T| = |T|$, so that $|T^*|^2 = TT^* = U|T||T^*|U^* = U|T|^2|T^*|U^* = (U|T||T^*|)^2$, then we have $|T^*| = U|T||T^*|$ because $U|T||T^*|$ is positive. Therefore $|T^*| = U|T||T^*|$. (ii) By (i), we have $T^* = U^*U|T|U^* = |T^*|U^* = T^*$ and $U^*x = 0$ if and only if $UU^*x = 0$ if and only if $|T^*|U^*x = 0$ by $N(U) = N(|T|)$ if and only if $T^*x = 0$ if and only if $TT^*x = 0$ if and only if $|T^*|x = 0$. Then $N(U^*) = N(|T^*|)$ and $U^*$ is also a partial isometry. So the proof of (ii) is complete.

**PROOF OF THEOREM 1.** In Theorem A, we replace $x_1$ by $|T|^{1/2}x_1$ and $x_k$ by $|T|^{1/2}U^*x_k$ for $k = 2, 3, \ldots, n$. Then we have the following by Theorem B:

$$(|T|^{1/2}x_1, |T|^{1/2}U^*x_k) = (U|T|x_1, x_k) = (Tx_1, x_k) \quad \text{for } k = 2, 3, \ldots, n,$$

$$(|T|^{1/2}U^*x_j, |T|^{1/2}U^*x_k) = (U|T^*|x_j, x_k) = (|T^*|x_j, x_k) \quad \text{for } j, k = 2, 3, \ldots, n.$$ 

By Theorem A and Theorem B, we have

$$0 \leq G_n \leq \| |T|^{1/2}x_1 \|^2 \| |T|^{1/2}U^*x_2 \|^2 \cdots \| |T|^{1/2}U^*x_n \|^2$$

$$= (|T|x_1, x_1)(|T^*|x_2, x_2) \cdots (|T^*|x_n, x_n).$$

$G_n = 0$ if and only if $|T|^{1/2}x_1, |T|^{1/2}U^*x_2, \ldots, |T|^{1/2}U^*x_n$ are linearly dependent (by Theorem A) if and only if $|T|x_1, |T^*|x_2, \ldots, |T^*|x_n$ are linearly dependent (by the positivity of $|T|^{1/2}$) if and only if $S_1 = \{ |T|x_1, T^*x_2, T^*x_3, \ldots, T^*x_n \}$ is a system of linearly dependent vectors (by Theorem B). Then

$$US_1 = \{ U|T|x_1, UT^*x_2, UT^*x_3, \ldots, UT^*x_n \}$$

is a system of linearly dependent vectors if and only if

$$S_2 = \{ Tx_1, T^*x_2, T^*x_3, \ldots, T^*x_n \}$$

is a system of linearly dependent vectors (by Theorem B).

Conversely assume that $S_2$ is a system of linearly dependent vectors. Then $U^*S_2 = \{ U^*Tx_1, U^*T^*x_2, U^*T^*x_3, \ldots, U^*T^*x_n \}$ is a system of linearly dependent vectors if and only if $S_1 = \{ |T|x_1, T^*x_2, T^*x_3, \ldots, T^*x_n \}$ is a system of linearly dependent vectors by Theorem B, so that $S_1$ is a system of linearly dependent vectors if and only if $S_2$ is a system of linearly dependent vectors. The proof of equality for the right-hand side follows from Theorem A and the argument stated above in the first half of the proof. So the proof of Theorem 1 is complete.

**PROOF OF THEOREM 2.** In Theorem A we replace $x_{2k}$ by $|T|^{1/2}U^*x_{2k}$ for $k = 1, 2, \ldots, n$ and $x_{2k-1}$ by $|T|^{1/2}x_{2k-1}$ for $k = 1, 2, \ldots, n$. Then by Theorem B we have

$$(|T|^{1/2}U^*x_{2j}, |T|^{1/2}U^*x_{2k}) = (U|T|U^*x_{2j}, x_{2k})$$

$$= (|T^*|x_{2j}, x_{2k}) \quad \text{for } j, k = 1, 2, \ldots, n,$$

$$(|T|^{1/2}x_{2j-1}, |T|^{1/2}U^*x_{2k}) = (U|T|x_{2j-1}, x_{2k})$$

$$= (Tx_{2j-1}, x_{2k}) \quad \text{for } j, k = 1, 2, \ldots, n.$$
By Theorem A and Theorem B, we have
\[ 0 \leq G_{2n} = \prod_{i=1}^{n} |T|^{1/2} x_i \prod_{i=1}^{n} |T|^{1/2} U^* x_i \prod_{i=1}^{n} |T|^{1/2} U^* x_i \prod_{i=1}^{n} |T|^{1/2} U^* x_i. \]

Since the proofs of the left-hand side and the right-hand side of the equality are given in the same way as in the proofs of Theorem 1, we omit them.

**Proof of Corollary 1.** The proof follows from the inequality in Theorem 1 or Theorem 2.

**References**


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