ENLARGEMENTS OF ALMOST OPEN MAPPINGS

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ABSTRACT. If the enlargement of a bounded linear operator has dense range, then the operator must be almost open.

Introduction. If $X$ is a normed space we write [1]

\[(0.1) \quad Q(X) = \frac{l_\infty(X)}{c_0(X)}\]

for the enlargement of $X$, and if $T \in BL(X, Y)$ is a bounded linear operator between normed spaces we write

\[(0.2) \quad Q(T) : Q(X) \to Q(Y)\]

for the operator induced by $T$, so that for each $x \in l_\infty(X)$ we have

\[(0.3) \quad Q(T)(x + c_0(X)) = (Tx) + c_0(Y).\]

Now we recall that

\[(0.4) \quad Q(T) \text{ one-one} \Rightarrow T \text{ bounded below} \Rightarrow Q(T) \text{ bounded below}\]

and [1, Theorem 4.1],

\[(0.5) \quad Q(T) \text{ almost open} \Rightarrow T \text{ almost open} \Rightarrow Q(T) \text{ open}.\]

It is the purpose of this note to improve (0.5) by confirming the conjecture (4.1.3) of [1].

1.1

THEOREM. If $T \in BL(X, Y)$ is a bounded linear operator between normed spaces then

\[(1.1.1) \quad Q(T) \text{ dense} \Rightarrow T \text{ almost open} \Rightarrow Q(T) \text{ open}.\]

PROOF. Suppose $\varphi : l_\infty \to \mathbb{C}$ is a bounded linear functional for which

\[(1.1.2) \quad c_0 \subseteq \varphi^{-1}(0),\]

then for each normed space $X$ we may define

\[(1.1.3) \quad \varphi_X^* : Q(X^\uparrow) \to Q(X)\]

by setting, for each $f \in l_\infty(X^\uparrow)$ and each $x \in l_\infty(X),$

\[(1.1.4) \quad \varphi_X^*([f])([x]) = \varphi(f_n(x_n));\]

here $X^\uparrow$ denotes the usual dual of the normed space $X,$ $[x] = x + c_0(X)$ and $[f] = f + c_0(X^\uparrow)$ are cosets, and $f_n(x_n) = a \in l_\infty$ where $a_n = f_n(x_n)$ for each $n \in \mathbb{N}.$ The reader should check, using (1.1.2), that the right-hand side of (1.1.4) depends

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only on the cosets \([f]\) and \([x]\) and that the linear mapping \(\varphi^*(f): Q(X)^+ \to C\) is bounded. Using the Hahn-Banach theorem we claim

\[ (1.1.5) \quad [0] \neq [f] \in Q(X^+) \Rightarrow \varphi(f(x)) \neq 0 \text{ for some } x \in l_\infty(X), \varphi \in (l_\infty/c_0)^+; \]

for if \(f \in l_\infty(X^+)\) is not in \(c_0(X^+)\), then \(\limsup_n \|f_n\| > 0\) and hence there is \(x \in l_\infty(X)\) for which \(\limsup_n |f_n(x_n)| > 0\), so that \(f(x) \in l_\infty\) is not in \(c_0\). Now by the Hahn-Banach theorem there is \(\varphi \in (l_\infty)^+\) satisfying (1.1.2) for which \(\varphi(f(x)) \neq 0\).

If \(T \in BL(X, Y)\) and \(\varphi \in (l_\infty)^+\) satisfies (1.1.2) then we claim

\[ (1.1.6) \quad Q(T)^+ \varphi_Y = \varphi_X \circ Q(T^+); \]

for this is just the associative property

\[ (1.1.7) \quad \varphi(g(x,Tx)) = \varphi((gT)(x,)) \text{ for each } x \in l_\infty(X), g \in l_\infty(Y^+). \]

We are ready to make our final claim: if \(T \in BL(X, Y)\) then

\[ (1.1.8) \quad Q(T)^+ \text{ one-one } \Rightarrow Q(T^+) \text{ one-one}. \]

Indeed suppose \(Q(T^+)\) is not one-one, so that there is \(g \in l_\infty(Y^+)\) for which

\[ (1.1.9) \quad gT \in c_0(X^+) \quad \text{and} \quad g \notin c_0(Y^+), \]

and then by (1.1.5) there is \(\varphi \in (l_\infty)^+\) and \(y \in l_\infty(Y)\) for which

\[ (1.1.10) \quad c_0 \subseteq \varphi^{-1}(0) \quad \text{and} \quad \varphi(g(y)) \neq 0; \]

but now

\[ (1.1.11) \quad Q(T)^+ (\varphi_Y[g]) = [0] \in Q(X)^+ \quad \text{and} \quad [0] \neq \varphi_Y[g] \in Q(Y)^+. \]

A familiar application of the Hahn-Banach separation theorem now gives (1.1.1): If \(T \in BL(X, Y)\) then

\[ (1.1.12) \quad Q(T) \text{ dense } \Rightarrow Q(T)^+ \text{ one-one } \Rightarrow Q(T^+) \text{ one-one}; \]

and

\[ (1.1.13) \quad Q(T^+) \text{ one-one } \Rightarrow T^+ \text{ bounded below } \Rightarrow T \text{ almost open}. \]

For an alternative proof of Theorem 1.1 we can use ultrafilters on \(\mathbb{N}\) instead of linear functionals on \(l_\infty/c_0\) [2].

REFERENCES