ENLARGEMENTS OF ALMOST OPEN MAPPINGS

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ABSTRACT. If the enlargement of a bounded linear operator has dense range, then the operator must be almost open.

Introduction. If \( X \) is a normed space we write \([1]\)

\[ Q(X) = l_\infty(X)/c_0(X) \]

for the enlargement of \( X \), and if \( T \in \text{BL}(X, Y) \) is a bounded linear operator between normed spaces we write

\[ Q(T): Q(X) \to Q(Y) \]

for the operator induced by \( T \), so that for each \( x \in l_\infty(X) \) we have

\[ Q(T)(x + c_0(X)) = (Tx) + c_0(Y). \]

Now we recall that

\[ Q(T) \text{ one-one} \Rightarrow T \text{ bounded below} \Rightarrow Q(T) \text{ bounded below} \]

and \([1, \text{Theorem 4.1}]\),

\[ Q(T) \text{ almost open} \Rightarrow T \text{ almost open} \Rightarrow Q(T) \text{ open}. \]

It is the purpose of this note to improve (0.5) by confirming the conjecture (4.1.3) of [1].

1.1

THEOREM. If \( T \in \text{BL}(X, Y) \) is a bounded linear operator between normed spaces then

\[ Q(T) \text{ dense} \Rightarrow T \text{ almost open} \Rightarrow Q(T) \text{ open}. \]

Proof. Suppose \( \varphi: l_\infty \to \mathbb{C} \) is a bounded linear functional for which

\[ c_0 \subseteq \varphi^{-1}(0), \]

then for each normed space \( X \) we may define

\[ \varphi_X^*: Q(X^\dagger) \to Q(X)^\dagger \]

by setting, for each \( f \in l_\infty(X^\dagger) \) and each \( x \in l_\infty(X) \),

\[ \varphi_X^*([f])([x]) = \varphi(f(x^\dagger)); \]

here \( X^\dagger \) denotes the usual dual of the normed space \( X \), \([x] = x + c_0(X)\) and \([f] = f + c_0(X^\dagger)\) are cosets, and \( f(x) = a \in l_\infty \) where \( a_n = f_n(x_n) \) for each \( n \in \mathbb{N} \). The reader should check, using (1.1.2), that the right-hand side of (1.1.4) depends

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only on the cosets \([f]\) and \([x]\) and that the linear mapping \(\varphi^\wedge_x([f]): Q(X)^\dagger \to \mathbb{C}\) is bounded. Using the Hahn-Banach theorem we claim

\[(1.1.5) \quad [0] \neq [f] \in Q(X^\dagger) \Rightarrow \varphi(f_\ast(x_\ast)) \neq 0 \text{ for some } x \in l_\infty(X), \varphi \in (l_\infty/c_0)^\dagger;\]

for if \(f \in l_\infty(X^\dagger)\) is not in \(c_0(X^\dagger)\), then \(\limsup_n\|f_n\| > 0\) and hence there is \(x \in l_\infty(X)\) for which \(\limsup_n|f_n(x_n)| > 0\), so that \(f_\ast(x_\ast) \in l_\infty\) is not in \(c_0\). Now by the Hahn-Banach theorem there is \(\varphi \in (l_\infty)^\dagger\) satisfying (1.1.2) for which \(\varphi(f_\ast(x_\ast)) \neq 0\).

If \(T \in BL(X, Y)\) and \(\varphi \in (l_\infty)^\dagger\) satisfies (1.1.2) then we claim

\[(1.1.6) \quad Q(T)^\dagger \circ \varphi^\wedge_Y = \varphi^\wedge_x \circ Q(T^\dagger);\]

for this is just the associative property

\[(1.1.7) \quad \varphi(g_\ast(Tx_\ast)) = \varphi((gT)_\ast(x_\ast)) \text{ for each } x \in l_\infty(X), g \in l_\infty(Y^\dagger).\]

We are ready to make our final claim: if \(T \in BL(X, Y)\) then

\[(1.1.8) \quad Q(T)^\dagger \text{ one-one } \Rightarrow Q(T^\dagger) \text{ one-one}.\]

Indeed suppose \(Q(T^\dagger)\) is not one-one, so that there is \(g \in l_\infty(Y^\dagger)\) for which

\[(1.1.9) \quad gT \in c_0(X^\dagger) \quad \text{and} \quad g \notin c_0(Y^\dagger),\]

and then by (1.1.5) there is \(\varphi \in (l_\infty)^\dagger\) and \(y \in l_\infty(Y)\) for which

\[(1.1.10) \quad c_0 \subseteq \varphi^{-1}(0) \quad \text{and} \quad \varphi(g_\ast(y_\ast)) \neq 0;\]

but now

\[(1.1.11) \quad Q(T)^\dagger(\varphi^\wedge_Y[g]) = [0] \in Q(X)^\dagger \quad \text{and} \quad [0] \neq \varphi^\wedge_Y[g] \in Q(Y)^\dagger.\]

A familiar application of the Hahn-Banach separation theorem now gives (1.1.1): If \(T \in BL(X, Y)\) then

\[(1.1.12) \quad Q(T) \text{ dense } \Rightarrow Q(T)^\dagger \text{ one-one } \Rightarrow Q(T^\dagger) \text{ one-one} \quad \text{and} \quad (1.1.13) \quad Q(T^\dagger) \text{ one-one } \Rightarrow T^\dagger \text{ bounded below } \Rightarrow T \text{ almost open}.\]

For an alternative proof of Theorem 1.1 we can use ultrafilters on \(\mathbb{N}\) instead of linear functionals on \(l_\infty/c_0\) [2].

REFERENCES


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