COMPARISON THEOREMS FOR SECOND ORDER DIFFERENTIAL SYSTEMS

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Abstract. Comparison theorems are proved for second order linear differential systems of the form \((R_i y')' + P_i y = 0\), where \(R_i\) and \(P_i\) are continuous \(n \times n\) matrices and \(R_i\) is invertible, \(i = 1, 2\).

Let \(R\) and \(P\) be \(n \times n\) matrices with real elements which are continuous and let \(R\) be invertible on an \(x\)-interval \([a, \omega]\). We shall consider the second-order vector differential equation

\[(E) \quad (R(x)y')' + P(x)y = 0.\]

If (E) has a nontrivial solution \(u\) satisfying \(u(b) = u'(c) = 0 [u'(b) = u(c) = 0]\) for some \(b\) and \(c\), \(a \leq b < c < \omega\), we define \(\eta(b) [\phi(b)]\) to be the infimum of \(\xi\), \(b \leq \xi < \omega\), such that there exists a nontrivial solution \(u\) of (E) satisfying \(u(b) = u'(\xi) = 0 [u'(b) = u(\xi) = 0]\). Otherwise, we put \(\eta(b) = \omega [\phi(b) = \omega]\). If \(\eta(b) < \omega [\phi(b) < \omega]\), then (E) has a nontrivial solution \(y\) such that \(y(b) = y'(\eta(b)) = 0 [y'(b) = y(\phi(b)) = 0]\). \(\phi(b)\) is called the right-hand focal point of \(b\). In recent years some authors have referred to \(\eta(b)\) as a focal point of \(b\); however, this appears to be inconsistent with the long-term usage of "focal" [13]. In Picone's terminology, \(\eta(b)\) is a right-hand pseudoconjugate of \(b\) and \(\phi(b)\) is a right-hand hemicontinuous to \(b\). We shall henceforth call \(\eta(b)\) the right-hand pseudoconjugate of \(b\).

Morse [11] was the first to obtain generalizations of the classical Sturm separation and comparison theorems for the second-order vector differential equations

\[(E_i) \quad (R_i(x)y')' + P_i(x)y = 0, \quad i = 1, 2,\]

where \(R_i\) and \(P_i\) are \(n \times n\) matrices with continuous and real elements and \(R_i\) is invertible on \([a, \omega]\), \(i = 1, 2\). Other comparison results of a different nature have been recently proved by Ahmad and Lazer [1–3] for the case \(R_i = I\), and also by others [6, 10, 14, 15] under various assumptions on \(R_i\) and \(P_i\). In [15] Tomastik also considered comparison theorems for the right-hand and left-hand focal points. It is to be noted that in most of these studies the case \(P_1 = P_2\) is specifically excluded.
Let \( \eta_i(b) \) be the right-hand pseudoconjugate [the right-hand focal point] of \( b \) for \( (E_i), \ i = 1, 2 \). The purpose of this paper is to present theorems comparing \( \eta_1(b) \) and \( \eta_2(c) \), where \( b \) and \( c \) are not necessarily equal. For the special case \( R_1 = R_2, P_1 = P_2 \), these results become "separation theorems," from which we can further deduce that \( \eta_i(x) \) is a nondecreasing function of \( x \).

The Riccati equation technique \([5, 8, 9, 12]\) adapted to the second-order system (E) is used to establish the main theorems.

**Theorem 1.** Let \( b \) be a point on the interval \([a, \omega)\). Every nontrivial vector solution \( y \) of (E) with \( y(b) = 0 \) has the property that \( y'(x) \not= 0, b < x < \omega \), if and only if the matrix Riccati system

\[
S' = R^{-1} + SPS, \quad S(b) = 0,
\]

has a solution on \([b, \omega)\).

**Proof.** Let \( Y \) be the solution of the matrix system

\[
(R(x)Y')' + P(x)Y = 0, \quad Y(b) = 0, \quad Y'(b) = I.
\]

To prove the necessity, let \( a \) be an arbitrary nonzero constant vector. Then \( y(x) = Y(x)a \) is a nontrivial solution of (E) with \( y(b) = 0 \). Since \( y'(x) = Y'(x)a \not= 0, b < x < \omega \), we see that the determinant of \( Y'(x) \) does not vanish on \([b, \omega)\). Thus, \( Y' \) is invertible on \([b, \omega)\). Since \( R \) is also invertible on \([b, \omega)\), so is \( RY' \). Consequently, \( S = Y(RY')^{-1} \) is defined and continuously differentiable on \([b, \omega)\) and \( S(b) = 0 \). Differentiating \( S \), we obtain \( S' = R^{-1} + SPS \), which proves that \( S \) is a solution of (MR) on \([b, \omega)\).

For \( n \times n \) matrices \( A = (a_{ij}) \) and \( B = (b_{ij}) \), we write \( A \geq B \) if \( a_{ij} \geq b_{ij}, i, j = 1, \ldots, n \), and we define

\[
\int_b^x A(t) \, dt = \left( \int_b^x a_{ij}(t) \, dt \right).
\]

In order to prove the sufficiency, we require the following lemma.

**Lemma 1.** The matrix Riccati equation (MR) has a unique solution \( S \) on \( J = [b, \eta(b)) \). The solution \( S \) is continuously differentiable and nontrivial; furthermore, it is nonnegative on \( J \) if

\[
R^{-1}(x) \geq 0, \quad P(x) \geq 0, \quad b \leq x < \eta(b).
\]

**Proof.** If \( R^{-1} = (r_{ij}), P = (p_{ij}), \) and \( S = (s_{ij}) \), the system (MR) is equivalent to the system of \( n^2 \) first-order equations

\[
s'_{ij} = t_{ij} + \sum_{k=1}^n s_{ik} \sum_{l=1}^n p_{kl}s_{lj}, \quad s_{ij}(b) = 0,
\]

\( i, j = 1, 2, \ldots, n \). Evidently, the above system may be cast into a vector equation of the form

\[
s' = f(x, s), \quad s(b) = 0,
\]
where \( s \) and \( f \) are \( n^2 \)-dimensional vectors. The vector-valued function \( f \) is continuous on \( D = \{(x, s): x \in J, |s| < \infty\} \); indeed, it is continuously differentiable on \( D \) as a function of \( s \). Therefore, \( f(x, s) \) satisfies a Lipschitz condition with respect to \( s \) on any compact and convex subset of \( D \) (see, e.g., [4, p. 142]) and there exists a unique solution \( s \in C' \) of (2) on some interval \([b, c], b < c < \eta(b)\) [7, p. 10]. Hence, the matrix Riccati system (MR) has a unique solution \( S \), continuously differentiable on the interval \([b, c]\). Let \( c_1 = \sup\{c: (MR) \text{ has a unique solution on } [b, c], b < c < \eta(b)\} \), and let \( S^* \) be the unique solution of (MR) on \([b, c_1]\). Since the derivative of every nontrivial vector solution \( y \) of (E) with \( y(b) = 0 \) does not vanish on \([b, \eta(b)]\), it follows from the necessity of Theorem 1 that (MR) has a solution, say \( S^0 \), on \([b, \eta(b)]\); thus, \( S^* = S^0 \) on \([b, c_1]\) by the uniqueness of solutions. If \( c_1 < \eta(b) \), we may assume that \( S^* \) is defined on \([b, c_1]\) (by setting \( S^*(c_1) = \lim_{x \to c_1} S^*(x) = S^0(c_1) \), if necessary). The solution \( S^* \) may then be continued to a right neighborhood \([c_1, c_1 + \varepsilon], \varepsilon > 0, \) of \( c_1 \) [7, p. 15]. This implies that (MR) has a unique solution on \([b, c_1 + \varepsilon], \varepsilon > 0, \) contrary to the choice of \( c_1 \). Therefore, \( c_1 = \eta(b) \) and (MR) has a unique solution \( S \) on \( J \).

The solution \( S \) is continuously differentiable because \( R^{-1} \) and \( P \) are continuous. Furthermore, \( S \) is nontrivial because \( R^{-1} \neq 0 \)---\( R^{-1} \) is invertible on \([b, \eta(b)]\) and it cannot have zero rows or zero columns at any point of \([b, \eta(b)]\)—and it may be obtained as the uniform limit of the successive approximations \( \{S_k\} \) defined recursively by the formula

\[
S_0(x) = 0, \\
S_{k+1}(x) = \int_b^x R^{-1}(t) \, dt + \int_b^x S_k(t) P(t) S_k(t) \, dt, \\
k = 0, 1, \ldots, \text{ on some interval } [b, d], b < d < \eta(b) (\text{see, e.g., [7, p. 12]}).
\]

Due to the inequalities (1), \( S_k \geq 0 \) on \([b, d]\), \( k = 0, 1, \ldots, \) and therefore the uniform limit \( S \geq 0 \) on \([b, d]\). Let \( d_1 = \sup\{d: S \geq 0 \text{ on } [b, d], b < d < \eta(b)\} \). Then \( S \geq 0 \) on \([b, d_1]\). We shall prove that \( d_1 = \eta(b) \). If \( d_1 < \eta(b) \), then \( 0 \leq S < \infty \) on \([b, d_1]\) by the continuity of \( S \). In this case, \( S \) may again be represented on some interval \([d_1, e], d_1 < e < \eta(b)\), as the uniform limit of the successive approximations

\[
S_0(x) = S(d_1) \geq 0, \\
S_{k+1}(x) = S(d_1) + \int_{d_1}^x R^{-1}(t) \, dt + \int_{d_1}^x S_k(t) P(t) S_k(t) \, dt, \\
k = 0, 1, \ldots, \text{Since } S_k \geq 0 \text{ on } [d_1, e], k = 0, 1, \ldots, \text{and } S \geq 0 \text{ on } [d_1, e]. \text{ We are thus led to the conclusion that } S \geq 0 \text{ on } [b, e], \text{ contrary to the choice of } d_1. \text{ Consequently, } d_1 = \eta(b) \text{ and } S \geq 0 \text{ on } [b, \eta(b)].
\]

Returning now to the proof of Theorem 1, we shall first prove that \(|Y'|, \text{ the determinant of } Y', \text{ does not vanish on } [b, \omega]\) if (MR) has a solution \( S \) on \([b, \omega]\). Since \(|Y'| \) is continuous and \(|Y'(\omega)| = 1 \text{ by (M)}\), \(|Y'| \) does not vanish on some right neighborhood \( N \) of the point \( b \), that is, \( Y' \) is invertible on \( N \). Since \( R \) is invertible, \( Y(RY')^{-1} \) is defined on \( N \) and satisfies (MR), as was shown earlier. Due to the uniqueness of solutions of the initial value problem (MR) proved in Lemma 1, we
have $S = Y(RY')^{-1}$ on $N$. Suppose that $|Y'|$ vanishes at some point on $[b, \omega)$: Let $\tilde{x}$ be the first point to the right of $b$ at which $|Y'|$ vanishes. Then there exists a nonzero constant vector $\beta$ such that $Y'(\tilde{x})\beta = 0$. On the interval $[b, \tilde{x})$ we have $S = Y(RY')^{-1}$, which may be written as $SRY' = Y$; this equality is indeed valid on $[b, \tilde{x})$ because $S, R, Y$ and $Y'$ are continuous on $[b, \tilde{x})$. In particular, $S(\tilde{x})R(\tilde{x})Y'(\tilde{x})\beta = Y(\tilde{x})\beta = 0$. But this is absurd since $w = Y\beta$ is a nontrivial solution of (E) and it cannot satisfy the condition $w(\tilde{x}) = w'(\tilde{x}) = 0$. Therefore, $|Y'|$ cannot vanish on $[b, \omega)$.

If $y$ is any nontrivial solution of (E) with $y(b) = 0$, then there exists a nonzero constant vector $\gamma$ such that $y = Y\gamma$. Evidently, $y' = Y'\gamma \neq 0$ on $[b, \omega)$ because $|Y'| \neq 0$ on $[b, \omega)$. This completes the proof.

Another result we need for proving comparison theorems is a version of Lemma 3.2 [12], strengthened for the matrix Riccati systems

$$\text{(MR)}_i \quad S' = R_i^{-1} + SP_i S, \quad S(b) = 0, \quad i = 1, 2.$$

**Lemma 2.** Let $R_i$ and $P_i$ be $n \times n$ matrices with continuous and real elements and let $R_i$ be invertible on an interval $[a, \omega)$, $i = 1, 2$. Assume that

$$0 \leq \int_b^x R_2^{-1}(t) \, dt \leq \int_b^x R_1^{-1}(t) \, dt, \quad 0 \leq P_2(x) \leq P_1(x), \quad b \leq x < \omega,$$

for some $b, a \leq b < \omega$. If there exists a nonnegative differentiable matrix $S$ defined on $[b, \omega)$ satisfying the matrix inequality

$$S' \geq R_1^{-1} + SP_1 S, \quad S(b) = S_b \geq 0,$$

then the matrix differential equation

$$T' = R_2^{-1} + TP_2 T, \quad T(b) = T_b, \quad S_b \geq T_b \geq 0,$$

has a continuous solution $T \leq S$ on $[b, \omega)$.

**Proof.** The existence of $T$ is proved by the iteration procedure

$$T_0(x) = S, \quad T_{k+1}(x) = T_b + \int_b^x R_2^{-1}(t) \, dt + \int_b^x T_k(t)P_2(t)T_k(t) \, dt,$$

$b \leq x \leq \omega$, $k = 0, 1, \ldots$ (cf. [12]). For $k = 0,$

$$0 \leq T_1(x) = T_b + \int_b^x R_2^{-1}(t) \, dt + \int_b^x S(t)P_2(t)S(t) \, dt$$

$$\leq S_b + \int_b^x R_1^{-1}(t) \, dt + \int_b^x S(t)P_1(t)S(t) \, dt \leq S(x) = T_0(x),$$

due to (3), (4), (5) and the nonnegativity of $S$; hence, $T_1$ is continuously differentiable and $0 \leq T_1 \leq T_0$ on $[b, \omega)$. From (6) we see that $T_{k+1} \geq 0$ if $T_k \geq 0$. Also, for $k = 0, 1, \ldots,$

$$T_{k+1}(x) - T_k(x) = \int_b^x \left[ T_k(t)P_2(t)T_k(t) - T_{k-1}(t)P_2(t)T_{k-1}(t) \right] \, dt,$$

where the integrand is nonpositive if $0 \leq T_k \leq T_{k-1}$. Therefore, $0 \leq T_{k+1} \leq T_k$ if $0 \leq T_k \leq T_{k-1}$. Since $0 \leq T_1 \leq T_0$, the sequence of continuously differentiable matrices $\{T_k\}$ decreases monotonically and is bounded below by zero. Furthermore,
the sequence is equicontinuous on any compact subinterval \( K \) of \([b, \omega)\). To show this, let \( \|A\| \) be the norm of an \( n \times n \) matrix \( A = (a_{ij}) \) defined by \( \|A\| = \sum_{i,j=1}^{n} |a_{ij}|. \) Let \( M > 0 \) be a constant such that \( \|R_2^{-1}\|, \|P_2\|, \) and \( \|T_k\|, k = 0, 1, \ldots, \) are all bounded by \( M \) on \( K \). From (6),

\[
T_{k+1}(x_2) - T_{k+1}(x_1) = \int_{x_1}^{x_2} R_2^{-1}(t) \, dt + \int_{x_1}^{x_2} T_k(t) P_2(t) T_k(t) \, dt,
\]

\( x_1, x_2 \in K, \, k = 0, 1, \ldots \). Thus,

\[
\|T_{k+1}(x_2) - T_{k+1}(x_1)\| \leq \int_{x_1}^{x_2} \|R_2^{-1}(t)\| \, dt + \int_{x_1}^{x_2} \|T_k(t) P_2(t) T_k(t)\| \, dt \leq (M + M^3) |x_2 - x_1|, \quad x_1, x_2 \in K,
\]

\( k = 0, 1, \ldots, \), and this implies that the sequence \( \{T_k\} \) is equicontinuous on \( K \). Since it is also uniformly bounded on \( K \}, \{T_k\} \) converges uniformly on \( K \). The uniform limit \( T = \lim_{k \to \infty} T_k \) is a continuous solution of (5) and \( T \leq T_0 = S \) on \( K \). Since this conclusion holds for every compact subinterval of \([b, \omega)\), it holds for \([b, \omega)\).

We are now ready to prove a comparison theorem for \( \eta_i(x) \), the right-hand pseudoconjugate function of \( (E_i) \), \( i = 1, 2 \), defined on \([a, \omega)\).

**Theorem 2.** Let \( b \) be a point on the interval \([a, \omega)\). If

\[
R_1^{-1}(x) > 0, \quad \int_{b}^{x} R_1^{-1}(t) \, dt \geq \int_{b}^{x} R_2^{-1}(t) \, dt \geq 0, \quad P_1(x) \geq P_2(x) \geq 0,
\]

\( b \leq x < \omega \), then \( \eta_1(b) \leq \eta_2(b) \). If the stronger condition

\[
R_1^{-1}(x) \geq R_2^{-1}(x) \geq 0, \quad P_1(x) \geq P_2(x) \geq 0,
\]

\( a \leq x < \omega \), holds, then \( \eta_1(b) \leq \eta_2(c), \, a \leq b \leq c < \omega \).

**Proof.** Every nontrivial solution \( y \) of \( (E_i) \) with \( y(b) = 0 \) has the property that \( y' \neq 0 \) on \([b, \eta_1(b)]) \). Hence, the corresponding matrix Riccati equation \( (MR_i) \) has a solution \( S \) on \([b, \eta_1(b)] \) by Theorem 1. The solution \( S \) is nontrivial and nonnegative on \([b, \eta_1(b)] \) by Lemma 1. According to (7) and Lemma 2, the matrix Riccati system \( (MR_2) \) associated with \( (E_2) \) has a continuous solution \( T \) on \([b, \eta_1(b)] \). Therefore, by Theorem 1, every nontrivial solution vector \( w \) of \( (E_2) \) with \( w(b) = 0 \) has the property that \( w' \neq 0 \) on \([b, \eta_1(b)] \); consequently, \( \eta_1(b) \leq \eta_2(b) \).

If (8) holds and \( c \) is an arbitrary point of \([a, \omega)\), then

\[
\int_{c}^{x} R_1^{-1}(t) \, dt \geq \int_{c}^{x} R_2^{-1}(t) \, dt \geq 0, \quad a \leq c \leq x < \omega.
\]

For \( a \leq b \leq c < \eta_1(b) \), \( (MR_1) \) has a nontrivial solution \( S \) which is continuous and nonnegative on \([b, \eta_1(b)] \) by Theorem 1 and Lemma 1. Applying Lemma 2 to the interval \([c, \eta_1(b)] \), we conclude that the system \( T' = R_2^{-1} + TP_2T, \, T(c) = 0 \), has a matrix solution \( T \) on \([c, \eta_1(b)] \). Again by Theorem 1, if \( \nu \) is any nontrivial solution of \( (E_2) \) with \( \nu(c) = 0 \), then \( \nu' \) does not vanish on \([c, \eta_1(b)] \). Therefore, \( \eta_1(b) \leq \eta_2(c) \), \( a \leq b \leq c < \eta_1(b) \).

If, on the other hand, \( \eta_1(b) \leq c < \omega \), it is obvious that \( \eta_1(b) \leq \eta_2(c) \). This completes the proof.
When we put \( R_1 = R_2 = R \) and \( P_1 = P_2 = P \) in Theorem 2—many comparison theorems for the second-order systems \((E_i)\), \( i = 1, 2 \), fail to hold for this case—we obtain the following “separation theorem”: If \( R \) is invertible, \( R^{-1} \geq 0 \), and \( P > 0 \) on \([a, \omega)\), then the equation \((E)\) has no nontrivial solution \( y \) such that \( y(x) = y'(x) = 0 \), \( b \leq x_1 < x_2 < \eta(b) \), for any \( b, a \leq b < \omega \). This result is equivalent to the statement that \( \eta(x) \) is a nondecreasing function of \( x \) on \([a, \omega)\).

Let \( \phi_i(x) \) be the right-hand focal point of \( x \) for the equation \((E_i)\), \( i = 1, 2 \). There are analogous comparison results for \( \phi_i(x) \), \( i = 1, 2 \), which we summarize below.

Let \( U \) be the solution of the matrix system
\[
(R(x)U')' + P(x)U = 0, \quad U(b) = I, \quad U'(b) = 0,
\]
for some \( b, a \leq b < \omega \). Put \( V = -RU'U^{-1} \). If every nontrivial solution \( y \) of \((E)\) with \( y'(b) = 0 \) does not vanish on \([b, \omega)\), then \( U \) is invertible on \([b, \omega)\). Thus, \( V \) is defined on \([b, \omega)\) and satisfies thereon
\[
(MR') \quad V' = P + VR^{-1}V, \quad V(b) = 0.
\]
This proves the necessity part of the following theorem.

**Theorem 3.** Suppose that \( R \) and \( P \) are \( n \times n \) matrices with continuous and real elements and that \( R \) is invertible on an interval \([a, \omega)\). Let \( b \) be a point on \([a, \omega)\). Every nontrivial solution vector \( y \) of \((E)\) with \( y'(b) = 0 \) does not vanish on \([b, \omega)\) if and only if the matrix Riccati system \((MR')\) has a solution on \([b, \omega)\).

Sufficiency of this theorem may be proved in a manner similar to the corresponding proof of Theorem 1, using the following analogue of Lemma 1.

**Lemma 3.** The matrix Riccati equation \((MR')\) has a unique solution on \([b, \phi(b))\), which is continuously differentiable. The solution is nontrivial if \( P \neq 0 \) and it is nonnegative on \([b, \phi(b))\) if \( R^{-1}(x) \geq 0 \), \( P(x) \geq 0 \), \( b \leq x < \phi(b) \).

Using Theorem 3, Lemma 2 (with \( P_i \) and \( R_i^{-1} \) interchanged in (3), (4) and (5), \( i = 1, 2 \)) and Lemma 3, we can similarly prove the following comparison theorem for \( \phi_i(x) \).

**Theorem 4.** If, for some \( b, a \leq b < \omega \),
\[
P_1(x) \geq 0, \quad \int_b^x P_1(t) \, dt \geq \int_b^x P_2(t) \, dt \geq 0, \quad R_1^{-1}(x) \geq R_2^{-1}(x) \geq 0,
\]
\( b \leq x < \omega \), then \( \phi_2(b) \geq \phi_1(b) \). Moreover, if
\[
P_1(x) \geq P_2(x) \geq 0, \quad R_1^{-1}(x) \geq R_2^{-1}(x) \geq 0,
\]
\( a \leq x < c \), then \( \phi_2(c) \geq \phi_1(b) \), \( a \leq b \leq c < \omega \).

Putting \( P_1 = P_2 = P \) and \( R_1 = R_2 = R \) in Theorem 4, we again obtain a “separation theorem”, which is equivalent to the statement that \( \phi(x) \) is a nondecreasing function of \( x \).
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