COMPARISON THEOREMS FOR SECOND ORDER
DIFFERENTIAL SYSTEMS

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Abstract. Comparison theorems are proved for second order linear differential systems of the form \((R_i y')' + P_i y = 0\), where \(R_i\) and \(P_i\) are continuous \(n \times n\) matrices and \(R_i\) is invertible, \(i = 1, 2\).

Let \(R\) and \(P\) be \(n \times n\) matrices with real elements which are continuous and let \(R\) be invertible on an \(x\)-interval \([a, \omega]\). We shall consider the second-order vector differential equation

\[ (E) \quad (R(x) y')' + P(x) y = 0. \]

If \((E)\) has a nontrivial solution \(v\) satisfying \(v(b) = v'(c) = 0\) \([v'(b) = v(c) = 0]\) for some \(b\) and \(c\), \(a \leq b < c < \omega\), we define \(\eta(b) [\phi(b)]\) to be the infimum of \(\xi\), \(b \leq \xi < \omega\), such that there exists a nontrivial solution \(u\) of \((E)\) satisfying \(u(b) = u'(\xi) = 0\) \([u'(b) = u(\xi) = 0]\). Otherwise, we put \(\eta(b) = \omega [\phi(b) = \omega]\). If \(\eta(b) < \omega [\phi(b) < \omega]\), then \((E)\) has a nontrivial solution \(y\) such that \(y(b) = y'(\eta(b)) = 0\) \([y'(b) = y(\phi(b)) = 0]\). \(\phi(b)\) is called the right-hand focal point of \(b\). In recent years some authors have referred to \(\eta(b)\) as a focal point of \(b\); however, this appears to be inconsistent with the long-term usage of "focal" \([13]\). In Picone's terminology, \(\eta(b)\) is a right-hand pseudoconjugate of \(b\) and \(\phi(b)\) is a right-hand hemiconjugate to \(b\). We shall henceforth call \(\eta(b)\) the right-hand pseudoconjugate of \(b\).

Morse \([11]\) was the first to obtain generalizations of the classical Sturm separation and comparison theorems for the second-order vector differential equations

\[ (E_i) \quad (R_i(x) y')' + P_i(x) y = 0, \quad i = 1, 2, \]

where \(R_i\) and \(P_i\) are \(n \times n\) matrices with continuous and real elements and \(R_i\) is invertible on \([a, \omega]\), \(i = 1, 2\). Other comparison results of a different nature have been recently proved by Ahmad and Lazer \([1-3]\) for the case \(R_i = I\), and also by others \([6, 10, 14, 15]\) under various assumptions on \(R_i\) and \(P_i\). In \([15]\) Tomastik also considered comparison theorems for the right-hand and left-hand focal points. It is to be noted that in most of these studies the case \(P_1 = P_2\) is specifically excluded.
Let $\eta_i(b) [\phi_i(b)]$ be the right-hand pseudoconjugate [the right-hand focal point] of $b$ for $(E_i), \ i=1,2$. The purpose of this paper is to present theorems comparing $\eta_1(b) [\phi_1(b)]$ and $\eta_2(c) [\phi_2(c)]$, where $b$ and $c$ are not necessarily equal. For the special case $R_1 = R_2, \ P_1 = P_2$, these results become “separation theorems,” from which we can further deduce that $\eta_i(x) [\phi_i(x)]$ is a nondecreasing function of $x$.

The Riccati equation technique [5, 8, 9, 12] adapted to the second-order system $(E)$ is used to establish the main theorems.

**THEOREM 1.** Let $b$ be a point on the interval $[a, \omega)$. Every nontrivial vector solution $y$ of $(E)$ with $y(b) = 0$ has the property that $y'(x) \neq 0, \ b \leq x < \omega$, if and only if the matrix Riccati system

$$(MR) \quad S' = R^{-1} + SPS, \quad S(b) = 0,$$

has a solution on $[b, \omega)$.

**Proof.** Let $Y$ be the solution of the matrix system

$$(M) \quad (R(x)Y')' + P(x)Y = 0, \quad Y(b) = 0, \quad Y'(b) = I.$$

To prove the necessity, let $a$ be an arbitrary nonzero constant vector. Then $y(x) \equiv Y(x)a$ is a nontrivial solution of $(E)$ with $y(b) = 0$. Since $y'(x) = Y'(x)a \neq 0, \ b \leq x < \omega$, we see that the determinant of $Y'(x)$ does not vanish on $[b, \omega)$. Thus, $Y'$ is invertible on $[b, \omega)$. Since $R$ is also invertible on $[b, \omega)$, so is $RY'$. Consequently, $S \equiv Y(RY')^{-1}$ is defined and continuously differentiable on $[b, \omega)$ and $S(b) = 0$. Differentiating $S$, we obtain $S' = R^{-1} + SPS$, which proves that $S$ is a solution of $(MR)$ on $[b, \omega)$.

For $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, we write $A \geq B$ if $a_{ij} \geq b_{ij}, \ i, j = 1, \ldots, n$, and we define

$$\int_{b}^{x} A(t) dt = \left( \int_{b}^{x} a_{ij}(t) dt \right).$$

In order to prove the sufficiency, we require the following lemma.

**LEMMA 1.** The matrix Riccati equation $(MR)$ has a unique solution $S$ on $J = [b, \eta(b))$. The solution $S$ is continuously differentiable and nontrivial; furthermore, it is nonnegative on $J$ if

$$1) \quad R^{-1}(x) \geq 0, \quad P(x) \geq 0, \quad b \leq x < \eta(b).$$

**Proof.** If $R^{-1} = (t_{ij}), \ P = (p_{ij}),$ and $S = (s_{ij})$, the system $(MR)$ is equivalent to the system of $n^2$ first-order equations

$$s_{ij} = t_{ij} + \sum_{k=1}^{n} \sum_{l=1}^{n} p_{kl}s_{lj}, \quad s_{ij}(b) = 0,$$

$i, j = 1, 2, \ldots, n$. Evidently, the above system may be cast into a vector equation of the form

$$2) \quad s' = f(x, s), \quad s(b) = 0,$$
where \( s \) and \( f \) are \( n^2 \)-dimensional vectors. The vector-valued function \( f \) is continuous on \( D = \{(x, s): x \in J, |s| < \infty\} \); indeed, it is continuously differentiable on \( D \) as a function of \( s \). Therefore, \( f(x, s) \) satisfies a Lipschitz condition with respect to \( s \) on any compact and convex subset of \( D \) (see, e.g., [4, p. 142]) and there exists a unique solution \( s \in C' \) of (2) on some interval \([b, c], b < c < \eta(b)\)[7, p. 10]. Hence, the matrix Riccati system (MR) has a unique solution \( S \), continuously differentiable on the interval \([b, c]\). Let \( c_1 = \sup\{c: (MR) \text{ has a unique solution on } [b, c], b < c < \eta(b)\} \), and let \( S^* \) be the unique solution of (MR) on \([b, c_1]\). Since the derivative of every nontrivial vector solution \( y \) of (E) with \( y(b) = 0 \) does not vanish on \([b, \eta(b)]\), it follows from the necessity of Theorem 1 that (MR) has a solution, say \( S^0 \), on \([b, \eta(b)]\); thus, \( S^* = S^0 \) on \([b, c_1]\) by the uniqueness of solutions. If \( c_1 < \eta(b) \), we may assume that \( S^* \) is defined on \([b, c_1]\) (by setting \( S^*(c_1) = \lim_{x \to c_1} S^*(x) = S^0(c_1) \), if necessary). The solution \( S^* \) may then be continued to a right neighborhood \([c_1, c_1 + \varepsilon], \varepsilon > 0 \), of \( c_1 \) [7, p. 15]. This implies that (MR) has a unique solution on \([b, c_1 + \varepsilon], \varepsilon > 0 \), contrary to the choice of \( c_1 \). Therefore, \( c_1 = \eta(b) \) and (MR) has a unique solution \( S \) on \( J \).

The solution \( S \) is continuously differentiable because \( R^{-1} \) and \( P \) are continuous. Furthermore, \( S \) is nontrivial because \( R^{-1} \neq 0 \) and \( R^{-1} \) is invertible on \([b, \eta(b)]\) and it cannot have zero rows or zero columns at any point of \([b, \eta(b)]\)—and it may be obtained as the uniform limit of the successive approximations \( \{S_k\} \) defined recursively by the formula

\[
S_0(x) = 0,
S_{k+1}(x) = \int_b^x R^{-1}(t) \, dt + \int_b^x S_k(t) P(t) S_k(t) \, dt,
\]

\( k = 0, 1, \ldots \), on some interval \([b, d], b < d < \eta(b)\) (see, e.g., [7, p. 12]). Due to the inequalities (1), \( S_k \geq 0 \) on \([b, d]\), \( k = 0, 1, \ldots \), and therefore the uniform limit \( S \geq 0 \) on \([b, d]\). Let \( d_1 = \sup\{d: S \geq 0 \text{ on } [b, d], b < d < \eta(b)\} \). Then \( S \geq 0 \) on \([b, d_1]\). We shall prove that \( d_1 = \eta(b) \). If \( d_1 < \eta(b) \), then \( 0 \leq S < \infty \) on \([b, d_1]\) by the continuity of \( S \). In this case, \( S \) may again be represented on some interval \([d_1, e], d_1 < e < \eta(b)\), as the uniform limit of the successive approximations

\[
S_0(x) = S(d_1) > 0,
S_{k+1}(x) = S(d_1) + \int_{d_1}^x R^{-1}(t) \, dt + \int_{d_1}^x S_k(t) P(t) S_k(t) \, dt,
\]

\( k = 0, 1, \ldots \). Since \( S_k \geq 0 \) on \([d_1, e]\), \( k = 0, 1, \ldots, S \geq 0 \) on \([d_1, e]\). We are thus led to the conclusion that \( S \geq 0 \) on \([b, e]\), contrary to the choice of \( d_1 \). Consequently, \( d_1 = \eta(b) \) and \( S \geq 0 \) on \([b, \eta(b)]\).

Returning now to the proof of Theorem 1, we shall first prove that \( |Y'| \), the determinant of \( Y' \), does not vanish on \([b, \omega)\) if (MR) has a solution \( S \) on \([b, \omega)\). Since \( |Y'| \) is continuous and \( |Y'(b)| = 1 \) by (M), \( |Y'| \) does not vanish on some right neighborhood \( N \) of the point \( b \), that is, \( Y' \) is invertible on \( N \). Since \( R \) is invertible, \( Y(RY')^{-1} \) is defined on \( N \) and satisfies (MR), as was shown earlier. Due to the uniqueness of solutions of the initial value problem (MR) proved in Lemma 1, we
have $S = Y(RY')^{-1}$ on $N$. Suppose that $|Y'|$ vanishes at some point on $[b, \omega)$: Let $\tilde{x}$ be the first point to the right of $b$ at which $|Y'|$ vanishes. Then there exists a nonzero constant vector $\beta$ such that $Y'(\tilde{x})\beta = 0$. On the interval $[b, \tilde{x})$ we have $S = Y(RY')^{-1}$, which may be written as $SRY' = Y$; this equality is indeed valid on $[b, \tilde{x})$ because $S, R, Y$ and $Y'$ are continuous on $[b, \tilde{x})$. In particular, $S(\tilde{x})R(\tilde{x})Y'(\tilde{x})\beta = Y(\tilde{x})\beta = 0$. But this is absurd since $w = Y\beta$ is a nontrivial solution of (E) and it cannot satisfy the condition $w(\tilde{x}) = w'(\tilde{x}) = 0$. Therefore, $|Y'|$ cannot vanish on $[b, \omega)$.

If $y$ is any nontrivial solution of (E) with $y(b) = 0$, then there exists a nonzero constant vector $\gamma$ such that $y = Y\gamma$. Evidently, $y' = Y\gamma \neq 0$ on $[b, \omega)$ because $|Y'| \neq 0$ on $[b, \omega)$. This completes the proof.

Another result we need for proving comparison theorems is a version of Lemma 3.2 [12], strengthened for the matrix Riccati systems

\begin{equation} \tag{MR_i} S' = R_i^{-1} + SP_i S, \quad S(b) = 0, \quad i = 1, 2. \end{equation}

**Lemma 2.** Let $R_i$ and $P_i$ be $n \times n$ matrices with continuous and real elements and let $R_i$ be invertible on an interval $[a, \omega)$, $i = 1, 2$. Assume that

\begin{equation} \tag{3} 0 \leq \int_b^x R_2^{-1}(t)\,dt \leq \int_b^x R_1^{-1}(t)\,dt, \quad 0 \leq P_2(x) \leq P_1(x), \quad b \leq x < \omega, \end{equation}

for some $b$, $a \leq b < \omega$. If there exists a nonnegative differentiable matrix $S$ defined on $[b, \omega)$ satisfying the matrix inequality

\begin{equation} \tag{4} S' \geq R_1^{-1} + SP_1 S, \quad S(b) = S_b \geq 0, \end{equation}

then the matrix differential equation

\begin{equation} \tag{5} T' = R_2^{-1} + TP_2 T, \quad T(b) = T_b, \quad S_b \geq T_b \geq 0, \end{equation}

has a continuous solution $T \leq S$ on $[b, \omega)$.

**Proof.** The existence of $T$ is proved by the iteration procedure

\begin{equation} \tag{6} T_0(x) = S, \quad T_{k+1}(x) = T_b + \int_b^x R_2^{-1}(t)\,dt + \int_b^x T_k(t)P_2(t)T_k(t)\,dt, \quad b \leq x \leq \omega, \quad k = 0, 1, \ldots \quad (cf. [12]). \end{equation}

For $k = 0$, we have

\[
0 \leq T_1(x) = T_b + \int_b^x R_2^{-1}(t)\,dt + \int_b^x S(t)P_2(t)S(t)\,dt \leq S_b + \int_b^x R_1^{-1}(t)\,dt + \int_b^x S(t)P_1(t)S(t)\,dt \leq S(x) = T_0(x),
\]

due to (3), (4), (5) and the nonnegativity of $S$; hence, $T_1$ is continuously differentiable and $0 \leq T_1 \leq T_0$ on $[b, \omega)$. From (6) we see that $T_{k+1} \geq 0$ if $T_k \geq 0$. Also, for $k = 0, 1, \ldots,$

\[
T_{k+1}(x) - T_k(x) = \int_b^x \left[ T_k(t)P_2(t)T_k(t) - T_{k-1}(t)P_2(t)T_{k-1}(t) \right]\,dt,
\]

where the integrand is nonpositive if $0 \leq T_k \leq T_{k-1}$. Therefore, $0 \leq T_{k+1} \leq T_k$ if $0 \leq T_k \leq T_{k-1}$. Since $0 \leq T_1 \leq T_0$, the sequence of continuously differentiable matrices $\{T_k\}$ decreases monotonically and is bounded below by zero. Furthermore,
the sequence is equicontinuous on any compact subinterval \( K \) of \([b, \omega)\). To show this, let \( \|A\| \) be the norm of an \( n \times n \) matrix \( A = (a_{ij}) \) defined by \( \|A\| = \sum_{i,j=1}^{n} |a_{ij}| \). Let \( M > 0 \) be a constant such that \( \|R_{2}^{1}\|, \|P_{2}\|, \) and \( \|T_{k}\|, k = 0, 1, \ldots, \) are all bounded by \( M \) on \( K \). From (6),

\[
T_{k+1}(x_{2}) - T_{k+1}(x_{1}) = \int_{x_{1}}^{x_{2}} R_{2}^{1}(t) \ dt + \int_{x_{1}}^{x_{2}} T_{k}(t) P_{2}(t) T_{k}(t) \ dt,
\]

\( x_{1}, x_{2} \in K, k = 0, 1, \ldots \). Thus,

\[
\|T_{k+1}(x_{2}) - T_{k+1}(x_{1})\| \lesssim \int_{x_{1}}^{x_{2}} \|R_{2}^{1}(t)\| \ dt + \int_{x_{1}}^{x_{2}} \|T_{k}(t) P_{2}(t) T_{k}(t)\| \ dt \lesssim (M + M^{3})|x_{2} - x_{1}|, \quad x_{1}, x_{2} \in K,
\]

\( k = 0, 1, \ldots, \) and this implies that the sequence \( \{T_{k}\} \) is equicontinuous on \( K \). Since it is also uniformly bounded on \( K \), \( \{T_{k}\} \) converges uniformly on \( K \). The uniform limit \( T = \lim_{k \to \infty} T_{k} \) is a continuous solution of (5) and \( T \leq T_{0} = S \) on \( K \). Since this conclusion holds for every compact subinterval of \([b, \omega)\), it holds for \([b, \omega)\).

We are now ready to prove a comparison theorem for \( \eta_{i}(x) \), the right-hand pseudoconjugate function of \((E_{i})\), \( i = 1, 2 \), defined on \([a, \omega)\).

**Theorem 2.** Let \( b \) be a point on the interval \([a, \omega)\). If

\[
R_{1}^{-1}(x) \geq 0, \quad \int_{h}^{x} R_{1}^{-1}(t) \ dt \geq \int_{h}^{x} R_{2}^{-1}(t) \ dt \geq 0, \quad P_{1}(x) \geq P_{2}(x) \geq 0,
\]

\( b \leq x < \omega, \) then \( \eta_{1}(b) \leq \eta_{2}(b) \). If the stronger condition

\[
R_{1}^{-1}(x) \geq R_{2}^{-1}(x) \geq 0, \quad P_{1}(x) \geq P_{2}(x) \geq 0,
\]

\( a \leq x < \omega, \) holds, then \( \eta_{1}(b) \leq \eta_{2}(c), a \leq b \leq c < \omega. \)

**Proof.** Every nontrivial solution \( y \) of \((E_{1})\) with \( y(b) = 0 \) has the property that \( y' \neq 0 \) on \([b, \eta_{1}(b)]\). Hence, the corresponding matrix Riccati equation \((MR_{1})\) has a solution \( S \) on \([b, \eta_{1}(b)]\) by Theorem 1. The solution \( S \) is nontrivial and nonnegative on \([b, \eta_{1}(b)]\) by Lemma 1. According to (7) and Lemma 2, the matrix Riccati system \((MR_{2})\) associated with \((E_{2})\) has a continuous solution \( T \) on \([b, \eta_{1}(b)]\). Therefore, by Theorem 1, every nontrivial solution vector \( w \) of \((E_{2})\) with \( w(b) = 0 \) has the property that \( w' \neq 0 \) on \([b, \eta_{1}(b)]\); consequently, \( \eta_{1}(b) \leq \eta_{2}(b) \).

If (8) holds and \( c \) is an arbitrary point of \([a, \omega)\), then

\[
\int_{c}^{x} R_{1}^{-1}(t) \ dt \geq \int_{c}^{x} R_{2}^{-1}(t) \ dt \geq 0, \quad a \leq c \leq x < \omega.
\]

For \( a \leq b \leq c < \eta_{1}(b) \), \((MR_{1})\) has a nontrivial solution \( S \) which is continuous and nonnegative on \([b, \eta_{1}(b)]\) by Theorem 1 and Lemma 1. Applying Lemma 2 to the interval \([c, \eta_{1}(b)]\), we conclude that the system \( T' = R_{2}^{-1} + TP_{2}T, T(c) = 0 \), has a matrix solution \( T \) on \([c, \eta_{1}(b)]\). Again by Theorem 1, if \( v \) is any nontrivial solution of \((E_{2})\) with \( v(c) = 0 \), then \( v' \) does not vanish on \([c, \eta_{1}(b)]\). Therefore, \( \eta_{1}(b) \leq \eta_{2}(c), a \leq b \leq c < \eta_{1}(b). \)

If, on the other hand, \( \eta_{1}(b) \leq c < \omega, \) it is obvious that \( \eta_{1}(b) \leq \eta_{2}(c). \) This completes the proof.
When we put $R_1 = R_2 = R$ and $P_1 = P_2 = P$ in Theorem 2—many comparison theorems for the second-order systems (E$_i$, $i = 1, 2$) fail to hold for this case—we obtain the following "separation theorem": If $R$ is invertible, $R^{-1} > 0$, and $P > 0$ on $[a, \omega)$, then the equation (E) has no nontrivial solution $y$ such that $y(x_1) = y'(x_2) = 0$, $b \leq x_1 < x_2 < \eta(b)$, for any $b$, $a \leq b < \omega$. This result is equivalent to the statement that $\eta(x)$ is a nondecreasing function of $x$ on $[a, \omega)$.

Let $\phi_i(x)$ be the right-hand focal point of $x$ for the equation (E$_i$, $i = 1, 2$. There are analogous comparison results for $\phi_i(x)$, $i = 1, 2$, which we summarize below.

Let $U$ be the solution of the matrix system

\[
(R(x)U')' + P(x)U = 0, \quad U(b) = I, \quad U'(b) = 0,
\]

for some $b$, $a \leq b < \omega$. Put $V = -RU'U^{-1}$. If every nontrivial solution $y$ of (E) with $y'(b) = 0$ does not vanish on $[b, \omega)$, then $U$ is invertible on $[b, \omega)$. Thus, $V$ is defined on $[b, \omega)$ and satisfies thereon

\[
(MR') \quad V' = P + VR^{-1}V, \quad V(b) = 0.
\]

This proves the necessity part of the following theorem.

**Theorem 3.** Suppose that $R$ and $P$ are $n \times n$ matrices with continuous and real elements and that $R$ is invertible on an interval $[a, \omega)$. Let $b$ be a point on $[a, \omega)$. Every nontrivial solution vector $y$ of (E) with $y'(b) = 0$ does not vanish on $[b, \omega)$ if and only if the matrix Riccati system $(MR')$ has a solution on $[b, \omega)$.

Sufficiency of this theorem may be proved in a manner similar to the corresponding proof of Theorem 1, using the following analogue of Lemma 1.

**Lemma 3.** The matrix Riccati equation $(MR')$ has a unique solution on $[b, \phi(b))$, which is continuously differentiable. The solution is nontrivial if $P \neq 0$ and it is nonnegative on $[b, \phi(b))$ if $R^{-1}(x) \geq 0$, $P(x) \geq 0$, $b \leq x < \phi(b)$.

Using Theorem 3, Lemma 2 (with $P_i$ and $R_i^{-1}$ interchanged in (3), (4) and (5), $i = 1, 2$) and Lemma 3, we can similarly prove the following comparison theorem for $\phi_i(x)$.

**Theorem 4.** If, for some $b$, $a \leq b < \omega$,

\[
P_1(x) \geq 0, \quad \int_b^x P_1(t) \, dt \geq \int_b^x P_2(t) \, dt \geq 0, \quad R_1^{-1}(x) \geq R_2^{-1}(x) \geq 0,
\]

$b \leq x < \omega$, then $\phi_2(b) \geq \phi_1(b)$. Moreover, if

\[
P_1(x) \geq P_2(x) \geq 0, \quad R_1^{-1}(x) \geq R_2^{-1}(x) \geq 0,
\]

$a \leq x < \omega$, then $\phi_2(c) \geq \phi_1(b)$, $a \leq b < c < \omega$.

Putting $P_1 = P_2 = P$ and $R_1 = R_2 = R$ in Theorem 4, we again obtain a "separation theorem", which is equivalent to the statement that $\phi(x)$ is a nondecreasing function of $x$. 
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