COMPARISON THEOREMS FOR SECOND ORDER DIFFERENTIAL SYSTEMS

W. J. KIM

Abstract. Comparison theorems are proved for second order linear differential systems of the form $\left( R_i y' \right)' + P_i y = 0$, where $R_i$ and $P_i$ are continuous $n \times n$ matrices and $R_i$ is invertible, $i = 1, 2$.

Let $R$ and $P$ be $n \times n$ matrices with real elements which are continuous and let $R$ be invertible on an $x$-interval $[a, \omega)$. We shall consider the second-order vector differential equation

$$(E) \quad \left( R(x) y' \right)' + P(x) y = 0.$$

If $(E)$ has a nontrivial solution $v$ satisfying $v(b) = v'(c) = 0$ [$v'(b) = v(c) = 0$] for some $b$ and $c$, $a \leq b < c < \omega$, we define $\eta(b)$ [$\phi(b)$] to be the infimum of $\xi$, $b \leq \xi < \omega$, such that there exists a nontrivial solution $u$ of $(E)$ satisfying $u(b) = u'(\xi) = 0$ [$u'(b) = u(\xi) = 0$]. Otherwise, we put $\eta(b) = \omega$ [$\phi(b) = \omega$]. If $\eta(b) < \omega$ [$\phi(b) < \omega$], then $(E)$ has a nontrivial solution $y$ such that $y(b) = y'(\eta(b)) = 0$ [$y'(b) = y(\phi(b)) = 0$]. $\phi(b)$ is called the right-hand focal point of $b$. In recent years some authors have referred to $\eta(b)$ as a focal point of $b$; however, this appears to be inconsistent with the long-term usage of “focal” [13]. In Picone’s terminology, $\eta(b)$ is a right-hand pseudoconjugate of $b$ and $\phi(b)$ is a right-hand hemiconjugate to $b$. We shall henceforth call $\eta(b)$ the right-hand pseudoconjugate of $b$.

Morse [11] was the first to obtain generalizations of the classical Sturm separation and comparison theorems for the second-order vector differential equations

$$(E_i) \quad \left( R_i(x) y' \right)' + P_i(x) y = 0, \quad i = 1, 2,$$

where $R_i$ and $P_i$ are $n \times n$ matrices with continuous and real elements and $R_i$ is invertible on $[a, \omega)$, $i = 1, 2$. Other comparison results of a different nature have been recently proved by Ahmad and Lazer [1–3] for the case $R_i = I$, and also by others [6, 10, 14, 15] under various assumptions on $R_i$ and $P_i$. In [15] Tomastik also considered comparison theorems for the right-hand and left-hand focal points. It is to be noted that in most of these studies the case $P_1 = P_2$ is specifically excluded.
Let $\eta_i(b) [\phi_i(b)]$ be the right-hand pseudoconjugate [the right-hand focal point] of $b$ for $(E_i), \ i = 1, 2$. The purpose of this paper is to present theorems comparing $\eta_1(b) [\phi_1(b)]$ and $\eta_2(c) [\phi_2(c)]$, where $b$ and $c$ are not necessarily equal. For the special case $R_1 = R_2, \ P_1 = P_2$, these results become "separation theorems," from which we can further deduce that $\eta_i(x) [\phi_i(x)]$ is a nondecreasing function of $x$.

The Riccati equation technique [5, 8, 9, 12] adapted to the second-order system (E) is used to establish the main theorems.

**Theorem 1.** Let $b$ be a point on the interval $[a, \omega)$. Every nontrivial vector solution $y$ of (E) with $y(b) = 0$ has the property that $y'(x) \neq 0, \ b \leq x < \omega$, if and only if the matrix Riccati system

\[
S' = R^{-1} + SPS, \quad S(b) = 0,
\]

has a solution on $[b, \omega)$.

**Proof.** Let $Y$ be the solution of the matrix system

\[
(R(x)Y')' + P(x)Y = 0, \quad Y(b) = 0, \quad Y'(b) = I.
\]

To prove the necessity, let $a$ be an arbitrary nonzero constant vector. Then $y(x) \equiv Y(x)a$ is a nontrivial solution of (E) with $y(b) = 0$. Since $y'(x) = Y'(x)a \neq 0, \ b \leq x < \omega$, we see that the determinant of $Y'(x)$ does not vanish on $[b, \omega)$. Thus, $Y'$ is invertible on $[b, \omega)$. Since $R$ is also invertible on $[b, \omega)$, so is $RY'$.

Consequently, $S \equiv Y(RY')^{-1}$ is defined and continuously differentiable on $[b, \omega)$ and $S(b) = 0$. Differentiating $S$, we obtain $S' = R^{-1} + SPS$, which proves that $S$ is a solution of (MR) on $[b, \omega)$.

For $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, we write $A \geq B$ if $a_{ij} \geq b_{ij}, \ i, j = 1, \ldots, n$, and we define

\[
\int_b^x A(t) \, dt = \left( \int_b^x a_{ij}(t) \, dt \right).
\]

In order to prove the sufficiency, we require the following lemma.

**Lemma 1.** The matrix Riccati equation (MR) has a unique solution $S$ on $J = [b, \eta(b))$. The solution $S$ is continuously differentiable and nontrivial; furthermore, it is nonnegative on $J$ if

\[
R^{-1}(x) \geq 0, \quad P(x) \geq 0, \quad b \leq x < \eta(b).
\]

**Proof.** If $R^{-1} = (t_{ij}), \ P = (p_{ij}),$ and $S = (s_{ij})$, the system (MR) is equivalent to the system of $n^2$ first-order equations

\[
s'_{ij} = t_{ij} + \sum_{k=1}^n s_{ik} \sum_{l=1}^n p_{kl}s_{lj}, \quad s_{ij}(b) = 0,
\]

$i, j = 1, 2, \ldots, n$. Evidently, the above system may be cast into a vector equation of the form

\[
s' = f(x, s), \quad s(b) = 0,
\]
where \( s \) and \( f \) are \( n^2 \)-dimensional vectors. The vector-valued function \( f \) is continuous on \( D = \{(x, s): x \in J, |s| < \infty \} \); indeed, it is continuously differentiable on \( D \) as a function of \( s \). Therefore, \( f(x, s) \) satisfies a Lipschitz condition with respect to \( s \) on any compact and convex subset of \( D \) (see, e.g., [4, p. 142]) and there exists a unique solution \( s \in C' \) of (2) on some interval \([b, c], b < c < \eta(b)\) [7, p. 10]. Hence, the matrix Riccati system (MR) has a unique solution \( S \), continuously differentiable on the interval \([b, c]\). Let \( c_1 = \sup \{ c: (MR) \) has a unique solution on \([b, c], b < c < \eta(b)\} \), and let \( S^* \) be the unique solution of (MR) on \([b, c_1]\). Since the derivative of every nontrivial vector solution \( y \) of (E) with \( y(b) = 0 \) does not vanish on \([b, \eta(b)]\), it follows from the necessity of Theorem 1 that (MR) has a solution, say \( S^0 \), on \([b, \eta(b)]\); thus, \( S^* = S^0 \) on \([b, c_1]\) by the uniqueness of solutions. If \( c_1 < \eta(b) \), we may assume that \( S^* \) is defined on \([b, c_1]\) (by setting \( S^*(c_1) = \lim_{x \to c_1} S^*(x) = S^0(c_1) \), if necessary). The solution \( S^* \) may then be continued to a right neighborhood \([c_1, c_1 + \epsilon], \epsilon > 0 \) of \( c_1 \) [7, p. 15]. This implies that (MR) has a unique solution on \([b, c_1 + \epsilon], \epsilon > 0 \), contrary to the choice of \( c_1 \). Therefore, \( c_1 = \eta(b) \) and (MR) has a unique solution \( S \) on \( J \).

The solution \( S \) is continuously differentiable because \( R^{-1} \) and \( P \) are continuous. Furthermore, \( S \) is nontrivial because \( R^{-1} \neq 0 - R^{-1} \) is invertible on \([b, \eta(b)]\) and it cannot have zero rows or zero columns at any point of \([b, \eta(b)]\)—and it may be obtained as the uniform limit of the successive approximations \( \{S_k\} \) defined recursively by the formula

\[
S_0(x) = 0, \\
S_{k+1}(x) = \int_b^x R^{-1}(t) \, dt + \int_b^x S_k(t) P(t) S_k(t) \, dt, \\
k = 0, 1, \ldots, \text{ on some interval } [b, d], \ b < d < \eta(b) \text{ (see, e.g., [7, p. 12]). Due to the inequalities (1), } S_k \geq 0 \text{ on } [b, d], \ k = 0, 1, \ldots, \text{ and therefore the uniform limit } S \geq 0 \text{ on } [b, d]. \text{ Let } d_1 = \sup \{ d: S \geq 0 \text{ on } [b, d], \ b < d < \eta(b) \}. \text{ Then } S \geq 0 \text{ on } [b, d_1]. \text{ We shall prove that } d_1 = \eta(b). \text{ If } d_1 < \eta(b), \text{ then } 0 \leq S < \infty \text{ on } [b, d_1] \text{ by the continuity of } S. \text{ In this case, } S \text{ may again be represented on some interval } [d_1, e], \ d_1 < e < \eta(b), \text{ as the uniform limit of the successive approximations}

\[
S_0(x) = S(d_1) \geq 0, \\
S_{k+1}(x) = S(d_1) + \int_{d_1}^x R^{-1}(t) \, dt + \int_{d_1}^x S_k(t) P(t) S_k(t) \, dt, \\
k = 0, 1, \ldots. \text{ Since } S_k \geq 0 \text{ on } [d_1, e], \ k = 0, 1, \ldots, S \geq 0 \text{ on } [d_1, e]. \text{ We are thus led to the conclusion that } S \geq 0 \text{ on } [b, e], \text{ contrary to the choice of } d_1. \text{ Consequently, } d_1 = \eta(b) \text{ and } S \geq 0 \text{ on } [b, \eta(b)].

Returning now to the proof of Theorem 1, we shall first prove that \(|Y'\), the determinant of \(Y'\), does not vanish on \([b, \omega]\) if (MR) has a solution \( S \) on \([b, \omega]\). Since \(|Y'\) is continuous and \(|Y'(b)| = 1 \text{ by (M), } |Y'| \) does not vanish on some right neighborhood \(N\) of the point \(b\), that is, \(Y'\) is invertible on \(N\). Since \(R\) is invertible, \(Y(RY')^{-1}\) is defined on \(N\) and satisfies (MR), as was shown earlier. Due to the uniqueness of solutions of the initial value problem (MR) proved in Lemma 1, we
have \( S = Y(RY')^{-1} \) on \( N \). Suppose that \( |Y'| \) vanishes at some point on \([b, \omega)\): Let \( \tilde{x} \) be the first point to the right of \( b \) at which \( |Y'| \) vanishes. Then there exists a nonzero constant vector \( \beta \) such that \( Y'(\tilde{x})\beta = 0 \). On the interval \([b, \tilde{x})\) we have \( S = Y(RY')^{-1} \), which may be written as \( SRY' = Y \); this equality is indeed valid on \([b, \tilde{x})\) because \( S, R, Y \) and \( Y' \) are continuous on \([b, \tilde{x})\). In particular, \( S(\tilde{x})R(\tilde{x})Y'(\tilde{x})\beta = Y(\tilde{x})\beta = 0 \). But this is absurd since \( w = Y\beta \) is a nontrivial solution of (E) and it cannot satisfy the condition \( w(\tilde{x}) = w'(\tilde{x}) = 0 \). Therefore, \( |Y'| \) cannot vanish on \([b, \omega)\).

If \( y \) is any nontrivial solution of (E) with \( y(b) = 0 \), then there exists a nonzero constant vector \( \gamma \) such that \( y = Y\gamma \). Evidently, \( y' = Y'\gamma \neq 0 \) on \([b, \omega)\) because \( |Y'| \neq 0 \) on \([b, \omega)\). This completes the proof.

Another result we need for proving comparison theorems is a version of Lemma 3.2 [12], strengthened for the matrix Riccati systems

\[ (MR_i) \quad S' = R_i^{-1} + SP_iS, \quad S(b) = 0, \quad i = 1, 2. \]

**Lemma 2.** Let \( R_i \) and \( P_i \) be \( n \times n \) matrices with continuous and real elements and let \( R_i \) be invertible on an interval \([a, \omega)\), \( i = 1, 2 \). Assume that

\[ (3) \quad 0 \leq \int_b^x R_i^{-1}(t) \, dt \leq \int_b^x R_1^{-1}(t) \, dt, \quad 0 \leq P_2(x) \leq P_1(x), \quad b \leq x < \omega, \]

for some \( b \), \( a \leq b < \omega \). If there exists a nonnegative differentiable matrix \( S \) defined on \([b, \omega)\) satisfying the matrix inequality

\[ (4) \quad S' \geq R_1^{-1} + SP_1S, \quad S(b) = S_h \geq 0, \]

then the matrix differential equation

\[ (5) \quad T' = R_2^{-1} + TP_2T, \quad T(b) = T_h, \quad S_h \geq T_h \geq 0, \]

has a continuous solution \( T \leq S \) on \([b, \omega)\).

**Proof.** The existence of \( T \) is proved by the iteration procedure

\[ (6) \quad T_0(x) = S, \quad T_{k+1}(x) = T_h + \int_b^x R_2^{-1}(t) \, dt + \int_b^x T_k(t)P_2(t)T_k(t) \, dt, \]

\( b \leq x \leq \omega \), \( k = 0, 1, \ldots \) (cf. [12]). For \( k = 0 \),

\[ 0 \leq T_1(x) = T_h + \int_b^x R_2^{-1}(t) \, dt + \int_b^x S(t)P_2(t)S(t) \, dt \leq S_h + \int_b^x R_1^{-1}(t) \, dt + \int_b^x S(t)P_1(t)S(t) \, dt \leq S(x) = T_0(x), \]

due to (3), (4), (5) and the nonnegativity of \( S \); hence, \( T_1 \) is continuously differentiable and \( 0 \leq T_1 \leq T_0 \) on \([b, \omega)\). From (6) we see that \( T_{k+1} \geq 0 \) if \( T_k \geq 0 \). Also, for \( k = 0, 1, \ldots \),

\[ T_{k+1}(x) - T_k(x) = \int_b^x [T_k(t)P_2(t)T_k(t) - T_{k-1}(t)P_2(t)T_{k-1}(t)] \, dt, \]

where the integrand is nonpositive if \( 0 \leq T_k \leq T_{k-1} \). Therefore, \( 0 \leq T_{k+1} \leq T_k \) if \( 0 \leq T_k \leq T_{k-1} \). Since \( 0 \leq T_1 \leq T_0 \), the sequence of continuously differentiable matrices \( \{ T_k \} \) decreases monotonically and is bounded below by zero. Furthermore,
the sequence is equicontinuous on any compact subinterval \( K \) of \([b, \omega)\). To show this, let \( \|A\| \) be the norm of an \( n \times n \) matrix \( A = (a_{ij}) \) defined by \( \|A\| = \sum_{i,j=1}^n |a_{ij}| \). Let \( M > 0 \) be a constant such that \( \|R_2^{-1}\|, \|P_2\|, \) and \( \|T_k\|, k = 0,1, \ldots \) are all bounded by \( M \) on \( K \). From (6),

\[
T_{k+1}(x_2) - T_{k+1}(x_1) = \int_{x_1}^{x_2} R_2^{-1}(t) \, dt + \int_{x_1}^{x_2} T_k(t)P_2(t)T_k(t) \, dt,
\]

\( x_1, x_2 \in K, \ k = 0,1, \ldots \). Thus,

\[
\|T_{k+1}(x_2) - T_{k+1}(x_1)\| \leq \int_{x_1}^{x_2} \|R^{-1}(t)\| |dt| + \int_{x_1}^{x_2} \|T_k(t)P_2(t)T_k(t)\| |dt| \leq (M + M^3)|x_2 - x_1|, \quad x_1, x_2 \in K,
\]

\( k = 0,1, \ldots \), and this implies that the sequence \( \{T_k\} \) is equicontinuous on \( K \). Since it is also uniformly bounded on \( K \), \( \{T_k\} \) converges uniformly on \( K \). The uniform limit \( T = \lim_{k \to \infty} T_k \) is a continuous solution of (5) and \( T \leq T_0 = S \) on \( K \). Since this conclusion holds for every compact subinterval of \([b, \omega)\), it holds for \([b, \omega)\).

We are now ready to prove a comparison theorem for \( \eta_i(x) \), the right-hand pseudoconjugate function of \( (E_i) \), \( i = 1,2 \), defined on \([a, \omega)\).

**Theorem 2.** Let \( b \) be a point on the interval \([a, \omega)\). If

\[
(7) \quad R_1^{-1}(x) > 0, \quad \int_h^x R_1^{-1}(t) \, dt \geq \int_h^x R_2^{-1}(t) \, dt \geq 0, \quad P_1(x) \geq P_2(x) \geq 0,
\]

\( b \leq x < \omega \), then \( \eta_1(b) \leq \eta_2(b) \). If the stronger condition

\[
(8) \quad R_1^{-1}(x) > R_2^{-1}(x) > 0, \quad P_1(x) \geq P_2(x) \geq 0,
\]

\( a \leq x < c < \omega \), holds, then \( \eta_1(b) \leq \eta_2(c) \), \( a \leq b \leq c < \omega \).

**Proof.** Every nontrivial solution \( y \) of \( (E_1) \) with \( y(b) = 0 \) has the property that \( y' \neq 0 \) on \([b, \eta_1(b))\). Hence, the corresponding matrix Riccati equation \( (MR_1) \) has a solution \( S \) on \([b, \eta_1(b))\) by Theorem 1. The solution \( S \) is nontrivial and nonnegative on \([b, \eta_1(b))\) by Lemma 1. According to (7) and Lemma 2, the matrix Riccati system \( (MR_2) \) associated with \( (E_2) \) has a continuous solution \( T \) on \([b, \eta_1(b))\). Therefore, by Theorem 1, every nontrivial solution vector \( w \) of \( (E_2) \) with \( w(b) = 0 \) has the property that \( w' \neq 0 \) on \([b, \eta_1(b))\); consequently, \( \eta_1(b) \leq \eta_2(b) \).

If (8) holds and \( c \) is an arbitrary point of \([a, \omega)\), then

\[
\int_c^x R_1^{-1}(t) \, dt \geq \int_c^x R_2^{-1}(t) \, dt \geq 0, \quad a \leq c \leq x < \omega.
\]

For \( a \leq b \leq c < \eta_1(b) \), \( (MR_1) \) has a nontrivial solution \( S \) which is continuous and nonnegative on \([b, \eta_1(b))\) by Theorem 1 and Lemma 1. Applying Lemma 2 to the interval \([c, \eta_1(b))\), we conclude that the system \( T' = R_2^{-1} + TP_2T \), \( T(c) = 0 \), has a matrix solution \( T \) on \([c, \eta_1(b))\). Again by Theorem 1, if \( \nu \) is any nontrivial solution of \( (E_2) \) with \( \nu(c) = 0 \), then \( \nu' \) does not vanish on \([c, \eta_1(b))\). Therefore, \( \eta_1(b) \leq \eta_2(c) \), \( a \leq b \leq c < \eta_1(b) \).

If, on the other hand, \( \eta_1(b) \leq c < \omega \), it is obvious that \( \eta_1(b) \leq \eta_2(c) \). This completes the proof.
When we put $R_1 = R_2 = R$ and $P_1 = P_2 = P$ in Theorem 2—many comparison theorems for the second-order systems $(E_i)$, $i = 1, 2$, fail to hold for this case—we obtain the following “separation theorem”: If $R$ is invertible, $R^{-1} \geq 0$, and $P \geq 0$ on $[a, \omega)$, then the equation $(E)$ has no nontrivial solution $y$ such that $y(x_1) = y'(x_2) = 0$, $b \leq x_1 \leq x_2 < \eta(b)$, for any $b$, $a \leq b < \omega$. This result is equivalent to the statement that $\eta(x)$ is a nondecreasing function of $x$ on $[a, \omega)$.

Let $\phi_i(x)$ be the right-hand focal point of $x$ for the equation $(E_i)$, $i = 1, 2$. There are analogous comparison results for $\phi_i(x)$, $i = 1, 2$, which we summarize below.

Let $U$ be the solution of the matrix system

$$
\begin{align*}
(R(x)U')' + P(x)U &= 0, \\
U(b) &= I, \\
U'(b) &= 0,
\end{align*}
$$

for some $b$, $a \leq b < \omega$. Put $V = -RU'U^{-1}$. If every nontrivial solution $y$ of $(E)$ with $y'(b) = 0$ does not vanish on $[b, \omega)$, then $U$ is invertible on $[b, \omega)$. Thus, $V$ is defined on $[b, \omega)$ and satisfies thereon

$$(MR') \quad V' = P + VR^{-1}V, \quad V(b) = 0.$$ 

This proves the necessity part of the following theorem.

**Theorem 3.** Suppose that $R$ and $P$ are $n \times n$ matrices with continuous and real elements and that $R$ is invertible on an interval $[a, \omega)$. Let $b$ be a point on $[a, \omega)$. Every nontrivial solution vector $y$ of $(E)$ with $y'(b) = 0$ does not vanish on $[b, \omega)$ if and only if the matrix Riccati system $(MR')$ has a solution on $[b, \omega)$.

Sufficiency of this theorem may be proved in a manner similar to the corresponding proof of Theorem 1, using the following analogue of Lemma 1.

**Lemma 3.** The matrix Riccati equation $(MR')$ has a unique solution on $[b, \phi(b))$, which is continuously differentiable. The solution is nontrivial if $P \neq 0$ and it is nonnegative on $[b, \phi(b))$ if $R^{-1}(x) \geq 0$, $P(x) \geq 0$, $b \leq x < \phi(b)$.

Using Theorem 3, Lemma 2 (with $P_i$ and $R_i^{-1}$ interchanged in (3), (4) and (5), $i = 1, 2$) and Lemma 3, we can similarly prove the following comparison theorem for $\phi_i(x)$.

**Theorem 4.** If, for some $b$, $a \leq b < \omega$,

$$
P_1(x) \geq 0, \quad \int_b^x P_1(t) \ dt \geq \int_b^x P_2(t) \ dt \geq 0, \quad R_1^{-1}(x) \geq R_2^{-1}(x) \geq 0,$$

$b \leq x < \omega$, then $\phi_2(b) \geq \phi_1(b)$. Moreover, if

$$
P_1(x) \geq P_2(x) \geq 0, \quad R_1^{-1}(x) \geq R_2^{-1}(x) \geq 0,$$

$a \leq x < \omega$, then $\phi_2(c) \geq \phi_1(b)$, $a \leq b \leq c < \omega$.

Putting $P_1 = P_2 = P$ and $R_1 = R_2 = R$ in Theorem 4, we again obtain a “separation theorem”, which is equivalent to the statement that $\phi(x)$ is a nondecreasing function of $x$. 

REFERENCES


DEPARTMENT OF APPLIED MATHEMATICS AND STATISTICS, STATE UNIVERSITY OF NEW YORK, STONY BROOK, NEW YORK 11794