ON THE PUNCTUAL AND LOCAL ERGODIC THEOREM
FOR NONPOSITIVE POWER BOUNDED OPERATORS
IN \( L^p[0, 1] \), \( 1 < p < + \infty \)

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Abstract. We show in this note that there exists a function \( f \in \cap_{1 < p < + \infty} L^p[0, 1] \) and for each \( p \) an isomorphism \( T: L^p \to L^p \) such that \( \sup_{n \in \mathbb{Z}} \| T^n \| < + \infty \) and \( T \) does not satisfy the punctual ergodic theorem.

We give also an example of a one-parameter semigroup \( (T_t, t \geq 0) \) of power bounded operators in each \( L^p (1 < p < + \infty) \) for which the assertion of the local ergodic theorem \( \lim_{t \to +} \int T_s f \, ds \) converge almost everywhere as \( t \to + \) for all \( f \in L^p \) fails to be true.

1. Introduction, definitions and notations. Let \((\Omega, \mathcal{A}, P)\) be a \( \sigma \) finite measure space and \( L^p(\Omega, \mathcal{A}, P) \) \( (1 < p < + \infty) \) the Banach space of complex valued measurable functions on \( \Omega \) such that \( \| f \|^p = \int |f(x)|^p \, d\mu < + \infty \). An operator \( T: L^p(\Omega, \mathcal{A}, P) \to L^p(\Omega, \mathcal{A}, P) \) satisfies the punctual ergodic theorem (p.e.t.) if for each \( f \in L^p(\Omega, \mathcal{A}, P) \)

\[
M_n(T)f = \frac{1}{n} \sum_{i=0}^{n-1} T^i f
\]

is a.e. convergent in \( \Omega \).

It is known that \( T \) satisfies the punctual ergodic theorem in the following cases: for \( 1 < p < + \infty \), \( p \neq 2 \) and \( T \) an isometry (for an invertible isometry, A. Ionescu Tulcea [1] and for a general isometry R. V. Chacon and S. A. McGrath [2]), for \( 1 < p < + \infty \) and \( T \) a positive contraction (M. A. Akcoglu [3]) and for \( 1 < p < + \infty \) and \( T \) an isomorphism such \( T \) and \( T^{-1} \) are both power bounded and positive (A. de la Torre [4]). D. L. Burkholder [5] constructed a contraction \( T: L^2 \to L^2 \) which does not satisfy the punctual ergodic theorem. Akcoglu and Sucheston [6] remarked that \( T \) can be geometrically dilated to an isometry on an \( L^2 \) space which does not satisfy the punctual ergodic theorem.

M. Feder [7] constructed for each \( 1 < p \leq 2 \) an isomorphism \( T: L^p[0, 2] \to L^p[0, 2] \), so that \( T \) and \( T^{-1} \) both are power bounded, and \( T \) does not satisfy the p.e.t. In [14] we have shown that for each \( 1 < p < + \infty \) there exists a power bounded operator \( T: L^p_{\mathbb{R}}[0, 1] \to L^p_{\mathbb{R}}[0, 1] \) such that \( T \) and \( T^* \) do not satisfy the p.e.t.

In this paper we show first that there exists for each \( p \), \( 1 < p < + \infty \), an isomorphism \( T \) such that \( T \) and \( T^{-1} \) are both power bounded and \( T \) does not satisfy
the p.e.t. This result is then extended to some sequences of operators defined by regular sequences $(\alpha_{nK})$, introduced in [13].

**Definition 1.** Let $(\alpha_{nK})$ be a sequence of positive numbers. $(\alpha_{nK})$ is said to be regular if

(i) $\forall n \in \mathbb{N}, \sum_{K=0}^{\infty} \alpha_{nK} = 1$,

(ii) $\forall z \in \mathbb{C}, |z| \leq 1, z \neq 1, |\sum_{K=0}^{\infty} \alpha_{nK} z^K| \to 0$,

(iii) $\forall n \in \mathbb{N}$, the radius of convergence $r_n$ of the series $\sum_{K=0}^{\infty} \alpha_{nK} z^K$ satisfies the condition $r_n > 1$.

Let $T_t$ ($t \geq 0$) be a strongly continuous one-parameter semigroup of bounded linear operators on $L^p$. This means that

(i) $T_t$ is a bounded linear operator on $L^p$ for any $t \geq 0$,

(ii) $T_{t+s} = T_t \circ T_s$ for any $t, s \geq 0$ and $f \in L^p$,

(iii) $\lim_{t \to 0} ||T_t - T_s|| = 0$ for any $f \in L^p$.

We note $S,f = \int_0^t T_s f ds$. Local ergodic theorems assert that $\lim_{t \to 0} (S,t)f$ exists a.e. for all $f \in L^p$. Local ergodic theorems have been studied by many authors (see [8]). We shall adapt the construction given by Akcoglu and Krengel in [9] to get an example of local ergodic divergence for power bounded operators in $L^p$ for each $p, 1 < p < +\infty$.

II. On the punctual ergodic theorem in $L^p[0,1]$, $1 < p < +\infty$. Let $(\phi_n^K)$ be Haar’s system. The functions $\phi_n^K$ are defined by the equations

$$\phi_n^K(t) = \phi_0^K(t) = 1 \quad \text{for } t \in [0,1]$$

and, for $m = 2^n + k$ with $1 \leq k \leq 2^n$, $n = 0, 1, \ldots,$

$$\phi_m^K(t) = \phi_n^K(t) = \begin{cases} +\sqrt{2^n} & \text{for } t \in \left[\frac{2K - 2}{2^{n-1}}, \frac{2K - 1}{2^{n+1}}\right], \\
 -\sqrt{2^n} & \text{for } t \in \left[\frac{2K - 1}{2^{n+1}}, \frac{2K}{2^{n+1}}\right], \\
 0 & \text{elsewhere.} \end{cases}$$

P. L. Ulyanov [10] has obtained the following result.

**Theorem 1.** There exists a function $f \in \cap_{1 < p < +\infty} L^p[0,1]$, $f = \sum_{i=1}^{\infty} a_i \phi_i$, and a permutation of the integers $\pi$ such that the series $\sum a_{\pi(i)} \phi_{\pi(i)}(t)$ diverges unboundedly on [0,1] almost everywhere.

The first proof given in [10] uses a quite difficult modification of Zahirski’s construction. In a recent publication [11] P. L. Ulyanov gives a simpler proof (in $L^2$). From this article and the arguments of the first paper we can get a simpler proof of this theorem. Olevskii [15] gave also (in $L^2$) another proof of this result. From this theorem we can get the theorem announced.

**Theorem 2.** There exists a function $f \in \cap_{1 < p < +\infty} L^p[0,1]$ and for each $p, 1 < p < +\infty$, an isomorphism $T$ such that $T$ and $T^{-1}$ are both power bounded and

$$M_n(T)f = \frac{1 + T + T^2 + \cdots + T^{n-1}}{n} f$$

does not converge almost surely.
Proof. Let $p$ be fixed, $1 < p < +\infty$. From Haar's system $(\phi_n^{(K)}) = (\phi_n)$ we can get an unconditional base of $L^p[0,1]$, $(\tilde{\phi}_n^{(K)}) = (\tilde{\phi}_n)$. Consider the permutation $\pi$ and $f$ of the previous theorem; $(\tilde{\phi}_{n(\pi)}^{(K)})$ is again an unconditional base of $L^p[0,1]$ and if $f = \sum a_i \tilde{\phi}_i$, then $f = \sum a_{\pi(i)} \tilde{\phi}_{\pi(i)}$.

By using Remark 1 in [7] we can get an isomorphism $T$ ($T$ and $T^{-1}$ are both power bounded) and a sequence $n_1 < n_2 < \cdots < n_K < \cdots$ such that $\|M_{n_K}(T) - Q_K\| < +\infty$, where $Q_K$ is the projection

$$Q_K \left( \sum_{i=1}^{\infty} b_i \tilde{\phi}_{n(i)} \right) = \sum_{i \geq K} b_i \tilde{\phi}_{n(i)}.$$

We can then deduce easily that the sequence $M_{n_K}(T)f$ does not converge almost surely on $[0,1]$. This result extends easily to the case of operators defined by a regular sequence $(a_n^{(K)})$ (see Definition 1).

Theorem 3. Let $(a_n^{(K)})$ be a regular sequence and consider the sequence of operators

$$R_n(T) = \sum_{K=0}^{\infty} \alpha_n^{(K)} T^K.$$

Then for each $p$, $1 < p < +\infty$, there exists an isomorphism $T: L^p \to L^p$ and $f \in L^p$ such that the sequence $R_n(T)f$ does not converge almost surely.

Proof. Consider the unconditional base $(\tilde{\phi}_{n(\pi)}^{(K)})$ of the previous proof and let $P_m$ be the projection on $\phi_{n(\pi)}$. As in [12] the operator $T$ will be diagonal, i.e., $T = \sum_{i=1}^{\infty} \lambda_i P_i$, where the $\lambda_i$ are complex numbers with $|\lambda_i| = 1$ and $\lambda_i \neq 1$, $\forall i$. Take $\lambda_1$, $|\lambda_1| = 1$ and $\lambda_1 \neq 1$. Choose $(\epsilon_i)$ such that $\epsilon_i > 0$ and $\sum \epsilon_i = 1$. There exist an integer $n_1$ and a real number $\delta_1$ such that $|R_{n_1}(\lambda_1)| < \epsilon_1$ and $|R_{n_1}(z) - 1| < \epsilon_1$ if $|z - 1| < \delta_1$, by the regular properties of the sequence $(a_n^{(K)})$.

If $\lambda_i$, $n_i$, $\delta_i$ have been chosen for $1 \leq i \leq j$, then we choose $\lambda_{j+1}$ such that $|\lambda_{j+1} - 1| < \delta_1$ for $1 \leq i \leq j$. Then we take $n_{j+1} > n_j$ such that $|R_{n_{j+1}}(\lambda_i)| < \epsilon_{j+1}$ for $1 \leq i \leq j + 1$ and $\delta_j > 0$ such that $|R_{n_{j+1}}(\lambda) - 1| < \epsilon_{j+1}$ when $|\lambda - 1| < \delta_{j+1}$.

The end of the proof is then the same as for the previous theorem.

Remark 4. (1) Theorem 2 shows that the Abel means of an isomorphism $T$ such that $T$ and $T^{-1}$ are power bounded in $L^p$ are not always convergent a.e. (take $\alpha_k = (1/n)(1 - 1/n)^K$).

(2) An analog proof has been given in the case of contractions of $L^2$ in [13].

III. On the local ergodic theorem in $L^p[0,1]$, $1 < p < +\infty$. Let $(\tilde{\phi}_{n(\pi)}^{(K)})$ be the unconditional base of the previous section, $V_K = I - Q_K$, and $p$ fixed, $1 < p < +\infty$. We have the following lemma, containing ideas from [9].

Lemma 5. There exist sequences $(\lambda_n)$, $n = 1, \ldots, (t_K)$, $K = 1, \ldots$, with $\lambda_n > 0$ and $0 < t_K \to 0$ such that $\Sigma \|V_K - (1/t_K)S_n\| < +\infty$.

Proof. Choose $(\epsilon_K)$ $\epsilon_K > 0$ such that $\Sigma \epsilon_K < +\infty$. Then take $\lambda_1 > 0$ arbitrarily and choose $0 < t_1 < 1$ so small that

$$\left| \frac{e^{i\lambda_1 t_1} - 1}{t_1 \lambda_1} - 1 \right| < \epsilon_1.$$
If $\lambda_1, \lambda_2, \ldots, \lambda_K$ and $t_1, \ldots, t_K$ are already chosen, choose $\lambda_{K+1} > 0$ sufficiently large so that

$$\left| \frac{e^{i\lambda_{K+1}t_m} - 1}{i\lambda_{K+1}t_m} \right| < \varepsilon_m \quad \text{for each } m = 1, 2, \ldots, K.$$ 

Then choose $s < t_{K+1} < 1/(K+1)$ so small that

$$\left| \frac{e^{i\lambda_n t_{K+1}} - 1}{i\lambda_n t_{K+1}} - 1 \right| < \varepsilon_{K+1}.$$ 

**Theorem 6.** Given $p$, $1 < p < +\infty$, there exists a function $f \in \cap_1 < p < +\infty L^p[0,1]$ and a one-parameter semigroup $(T_t, t \geq 0)$ of power bounded operators in $L^p[0,1]$ for which $(1/t)^{1/p} T_t f \, dt$ does not converge a.e. as $t \to 0_+$.

**Proof.** Let $p$ fixed, $1 < p < +\infty$. Consider the one-parameter semigroup $T_t$ defined by

$$T_t \left( \sum_{n=1}^{\infty} \alpha_n \tilde{\phi}_n(n) \right) = \sum_{n=1}^{\infty} e^{i\lambda_n t} \alpha_n \tilde{\phi}_n(n)$$

where the $\lambda_n$ are those of the previous lemma and $\pi$ the permutation in Theorem 1. Then we have $T_0 = I$ (identity).

(i) for each $t, s, T_t \circ T_s = T_{t+s}$,

(ii) $\lim_{s \to t} \|T_s f - T_t f\|_p = 0$ $\forall t > 0$ and $\forall f \in L^p$,

(iii) $S_t f = \sum_{n=1}^{\infty} (1/i\lambda_n) \left( \exp(-i\lambda_n t) - 1 \right) \alpha_n \tilde{\phi}_n(n)$ ($t > 0$).

It is now easy to see that $(S_K/t_K)f$ does not converge almost everywhere for the function $f$ of Theorem 1.

**Remark 7.** It is known that for $T_0 = I$ and any semigroup of positive bounded linear operators on $L^p$ with $P(\Omega) < +\infty$ we can get the local ergodic theorem (see [8]).

**References**


14. ___, *Sur les opérateurs à puissances bornées et le théorème ergodique ponctuel dans $L^p[0,1]$, $1 < p < +\infty$* (a paraître).


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