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GROUND STATES AND SMALL PERTURBATIONS FOR C^*-DYNAMICAL SYSTEMS

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ABSTRACT. Existence of ground states for C^*-dynamical systems is stable under perturbations with relative bound < 1.

Let $(A, \alpha, \mathbf{R})$ be a C^*-dynamical system, δ its generator with domain $\mathcal{D}(\delta)$ and δ' a *-derivation in A with the same domain as δ, and assume that A has an identity. Then δ' is automatically relatively bounded with respect to δ [**10**], and if, furthermore, the relative bound of δ' with respect to δ (briefly, δ-bound) is smaller than one, then $\delta + \delta'$ is the generator of a C^*-dynamical system on A [**1**].

A state ϕ of A is said to be a ground state for α if $-i\phi(x^*\delta(x)) \geq 0$ for any x in $\mathcal{D}(\delta)$, or, equivalently, if ϕ is α-invariant and the selfadjoint operator H_ϕ is positive, where H_ϕ denotes the generator of the α-covariant unitary representation associated with ϕ. If α has a ground state, then so does any boundedly perturbed system of α [**5**]. Batty [**2**] showed, in the above situation, that existence of ground states is stable under perturbations with δ-bound < 1, or slightly more general small perturbations, provided that A is of type I. For a general C^*-algebra and a perturbation $\lambda\delta'$ with sufficiently small $|\lambda|$, it was recently proved by Kurose [**9**], by virtue of Batty's method in [**2**] and Kishimoto's results in [**8**]. We shall show this when δ-bound < 1. Its proof is similar to his and its key point is the calculation of relative bounds.

The following perturbation is slightly more general than perturbations with δ-bound < 1: There exist nonnegative constants a and $b < 1$ such that for any x in $\mathcal{D}(\delta)$,

$$(*) \qquad \|\delta'(x)\| \leq a\|x\| + b(\|\delta(x)\| + \|(\delta + \delta')(x)\|).$$

Stability of many properties, for example, closability, the property of generating dynamical system and existence of ground states under such perturbations, may be reduced to that under relatively bounded perturbations, in virtue of an iterating technique (see e.g. [**7**, p. 191]).

THEOREM. *Let $(A, \alpha, \mathbf{R})$ be a C^*-dynamical system with generator δ which has a ground state, and assume that A has an identity. Let δ' be a *-derivation in A with the same domain $\mathcal{D}(\delta)$ as δ.*

If $(*)$ is satisfied for some $b < 1$, then the perturbed system $(A, e^{t(\delta+\delta')}, \mathbf{R})$ also has a ground state.

First we show the following lemma.

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LEMMA. *Let α be a σ-weakly continuous one-parameter group of *-automorphisms of a von Neumann algebra M and δ its generator with domain $\mathcal{D}(\delta)$. Let A be a σ-weakly dense C^*-subalgebra of M such that $(1+\delta)(A \cap \mathcal{D}(\delta)) \supset A$.*

If δ' is a σ-weakly closable *-derivation in M with the same domain as δ, then the δ-bound of δ' coincides with the $\delta|A \cap \mathcal{D}(\delta)$-bound of $\delta'|A \cap \mathcal{D}(\delta)$.

PROOF. Assume that for some nonnegative numbers a and b and for any x in $A \cap \mathcal{D}(\delta)$,

$$\|\delta'(x)\| \leq a\|x\| + b\|\delta(x)\|.$$

If M_2 denotes a 2×2 full matrix algebra, then $\delta \otimes 1$ is the generator of $\alpha \otimes 1$ on $M \otimes M_2$ with domain $\mathcal{D}(\delta) \otimes M_2$. Let x be an element of $\mathcal{D}(\delta)$ with norm one, so that $\left(\begin{smallmatrix} 0 & x^* \\ x & 0 \end{smallmatrix}\right)$ is a selfadjoint element of $\mathcal{D}(\delta \otimes 1)$ with norm one, where $M \otimes M_2$ is identified with $M_2(M)$. Then, by the Kaplansky density theorem, $(1+\delta\otimes 1)\left(\left(\begin{smallmatrix} 0 & x^* \\ x & 0 \end{smallmatrix}\right)\right)$ belongs to the strong closure of the selfadjoint part of the ball of A with radius

$$\left\|(1+\delta\otimes 1)\left(\begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}\right)\right\|.$$

Let f be in $C^1(\mathbf{R})$ with compact support such that $f(t) = t$ on $[-1,1]$ and $f'^{\frown} \in L^1$, where $f'^{\frown}$ denotes the Fourier transform of the derivative f'. If $y \in A \otimes M_2$, $y^* = y$, $\|y\| \leq \|(1+\delta\otimes 1)\left(\left(\begin{smallmatrix} 0 & x^* \\ x & 0 \end{smallmatrix}\right)\right)\|$ and $z = (1+\delta\otimes 1)^{-1}y$, then it follows from [**5**, 3.2.32], that

$$f(z) = (2\pi)^{-1/2} \int dt f^{\frown}(t) e^{itz} \in \mathcal{D}(\delta \otimes 1) \cap (A \otimes M_2),$$

$$(\delta \otimes 1)(f(z)) = (2\pi)^{-1/2} \int dt f'^{\frown}(t) \int_0^1 ds\, e^{istz} (\delta \otimes 1)(z) e^{i(1-s)tz}$$

and

$$zf'(z) = (2\pi)^{-1/2} \int dt f'^{\frown}(t) \int_0^1 ds\, e^{istz} z e^{i(1-s)tz}.$$

$\delta'(1+\delta)^{-1}$ is σ-weakly closed and bounded, and so σ-weakly continuous, that is, the mapping $(x, \delta(x)) \mapsto \delta'(x)$ is σ-weakly continuous. Therefore, since $(1+\delta\otimes 1)^{-1}$ is strongly continuous on bounded sets, as y tends strongly to $(1+\delta\otimes 1)\left(\left(\begin{smallmatrix} 0 & x^* \\ x & 0 \end{smallmatrix}\right)\right)$, $f(z)_{21}$ tends strongly to $f\left(\left(\begin{smallmatrix} 0 & x^* \\ x & 0 \end{smallmatrix}\right)\right)_{21} = x$ and $\delta(f(z)_{21})$ to $\delta(x)$, hence $\delta'(f(z)_{21})$ tends σ-weakly to $\delta'(x)$. The above equalities imply the

following inequalities:

$$\begin{aligned}
\|\delta'(f(z)_{21})\| &\le a\|f(z)_{21}\| + b\|\delta(f(z)_{21})\| \\
&\le a\|f(z)\| + b\|\delta \otimes 1(f(z))\| \\
&\le a\|f(z)\| + b\|zf'(z)\| + b\|zf'(z) + \delta \otimes 1(f(z))\| \\
&\le a\|f(z)\| + b\|zf'(z)\| + b(2\pi)^{-1/2}\|f'\widehat{\ }\|_1\|y\| \\
&\le a\|f(z)\| + b\|zf'(z)\| + b(2\pi)^{-1/2}\|f'\widehat{\ }\|_1 \\
&\quad \times \left\|(1+\delta\otimes 1)\left(\begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}\right)\right\| \\
&\le (a\|f\|_\infty + b\sup_t |tf'(t)| + b(2\pi)^{-1/2}\|f'\widehat{\ }\|_1)\|x\| \\
&\quad + b(2\pi)^{-1/2}\|f'\widehat{\ }\|_1\|\delta(x)\|.
\end{aligned}$$

Since $\delta'(f(z)_{21})$ tends σ-weakly to $\delta'(x)$ as y tends strongly to $(1+\delta\otimes 1)(\left(\begin{smallmatrix} 0 & x^* \\ x & 0 \end{smallmatrix}\right))$, we have

$$\begin{aligned}
\|\delta'(x)\| \le &\left(a\|f\|_\infty + b\sup_t |tf'(t)| + b(2\pi)^{-1/2}\|f'\widehat{\ }\|_1\right)\|x\| \\
&+ b(2\pi)^{-1/2}\|f'\widehat{\ }\|_1\|\delta(x)\|.
\end{aligned}$$

Replacing x by $\|x\|^{-1}x$, this remains valid for any x in $\mathcal{D}(\delta)$.

It remains to choose a function f, for any $\varepsilon > 0$, such that $(2\pi)^{-1/2}\|f'\widehat{\ }\|_1 \le 1+\varepsilon$. There exists a continuous even function g_1 with compact support such that $g_1(t) = 1$ on $[-1,1]$ and $(2\pi)^{-1/2}\|g_1\widehat{\ }\|_1 \le 1 + 2^{-1}\varepsilon$ (see e.g. [**6**]). Define g_2 as follows:

$$g_2 = (2\pi)^{1/2}8^{-1}\varepsilon c^{-1/2} I_{[-c,c]}\widehat{\ } * I_{[-1,1]}\widehat{\ },$$

where $I_{[-c,c]}$ and $I_{[-1,1]}$ denote the indicator functions of $[-c,c]$ and $[-1,1]$, respectively, and $c = 4\varepsilon^{-2}(\int_0^\infty dt\, g_1(t))^2$, so that $\int_0^\infty dt\, g_1(t) = \int_{-\infty}^\infty dt\, g_2(t)$. We then have

$$\begin{aligned}
(2\pi)^{-1/2}\|g_2\widehat{\ }\|_1 &= 8^{-1}\varepsilon c^{-1/2}\|I_{[-c,c]}\widehat{\ } I_{[-1,1]}\widehat{\ }\|_1 \\
&\le 8^{-1}\varepsilon c^{-1/2}\|I_{[-c,c]}\widehat{\ }\|_2\|I_{[-1,1]}\widehat{\ }\|_2 \\
&= 8^{-1}\varepsilon c^{-1/2}\|I_{[-c,c]}\widehat{\ }\|_2\|I_{[-1,1]}\widehat{\ }\|_2 = 4^{-1}\varepsilon.
\end{aligned}$$

Define f as follows:

$$f(t) = \int_{-\infty}^t dt(g_1(s) - g_2(s-d) - g_2(s+d)),$$

where d is a number such that the supports of g_1 and $g_2(\cdot - d)$ are disjoint. It is then obvious that f is as desired. We thus complete the proof of the lemma.

PROOF OF THE THEOREM. We may assume that the δ-bound of δ' is smaller than one. If α has a ground state, then α has a pure ground state [**5**]. Let ϕ be a pure ground state for α, so that ϕ is α-invariant and induces an irreducible α-covariant representation $(\pi, e^{itH}, \mathcal{H}, \xi)$ and a *-derivation δ_π in $\pi_\phi(A)$. Since δ' has the same domain as δ, π_ϕ induces a *-derivation with domain $\pi_\phi(\mathcal{D}(\delta))$, $\pi_\phi(x) \mapsto \pi_\phi(\delta'(x))$,

whose δ_π-bound does not exceed the δ-bound of δ' [**1**]. Therefore we may assume without loss of generality that A is weakly dense in $\mathcal{B}(\mathcal{H})$, α_t is implemented by e^{itH} and H is a selfadjoint and positive operator such that for any x in $\mathcal{D}(\delta)$,

$$iHx\xi = \delta(x)\xi \quad \text{and} \quad \delta(x) = i\overline{[H,x]},$$

where ξ is a cyclic vector in $\mathcal{H}$ with norm one and $H\xi = 0$. Denote by $\overline{\delta}$ the generator of $\operatorname{Ad} e^{itH}$, the σ-weak closure of δ. Since δ' is σ-weakly closable [**8**], $\overline{\delta'}(1+\overline{\delta})^{-1}$ is σ-weakly closed and bounded, and so everywhere defined, that is, $\mathcal{D}(\overline{\delta'}) \supset \mathcal{D}(\overline{\delta})$. It therefore follows from the Lemma that the $\overline{\delta}$-bound of $\overline{\delta'}|\mathcal{D}(\overline{\delta})$ coincides with the δ-bound of δ', and hence is smaller than one. Then we can complete the proof, in the same way as [**2**, **9**]. If p is the projection onto ξ, then there exists a selfadjoint element k in A with $k\xi = i[\overline{\delta'}(p), p]\xi$, because of the irreducibility of A. By an easy computation we then have that $(\overline{\delta'} + \delta_{ik})(p) = 0$, so that a selfadjoint operator K is well defined by

$$iKx\xi = (\delta' + \delta_{ik})(x)\xi, \qquad x \in \mathcal{D}(\delta).$$

Noticing $\|xp\| = \|x\xi\|$ and $(\overline{\delta'} + \delta_{ik})(p) = 0$, it is not difficult to show that the H-bound of K is smaller than one [**4**, **9**], so that $H + K$ is lower semibounded by the Kato-Rellich theorem [**7**]. Then the proof of stability of ground states under bounded perturbations in [**5**, 5.4.14] assures that $e^{t(\delta+\delta'+\delta_{ik})}$, and hence the boundedly perturbed system $e^{t(\delta+\delta')}$, have ground states. We thus complete the proof of the Theorem.

Under the same assumptions as in the Theorem, if there exists a faithful family of irreducible α-invariant representations of A, then the closure $\overline{\delta + \delta'}$ is a generator [**8**] and we assert that $e^{t(\overline{\delta+\delta'})}$ has a ground state if the δ-bound of $\delta \leq 1$. For, choosing a sequence (λ_n) of positive numbers converging to 1 such that the δ-bound of $\lambda_n\delta'$ is smaller than one, it follows from the Theorem that each $e^{t(\delta+\lambda_n\delta')}$ has a ground state ϕ_n. If ϕ is a $\sigma(A^*, A)$-cluster point of (ϕ_n), then ϕ is obviously a ground state for $e^{t(\overline{\delta+\delta'})}$. We thus have the following corollary.

COROLLARY. *Let $(A, \alpha, \mathbf{R})$, δ and δ' be as in the Theorem. Furthermore assume that $\overline{\delta + \delta'}$ is a generator, which is satisfied if, in particular, A is simple.*

If the δ-bound of δ' is smaller than or equal to one, then $e^{t(\overline{\delta+\delta'})}$ has a ground state.

Let $(A, \alpha, \mathbf{R})$, δ and δ' be as in the Theorem, and assume that A is a UHF algebra. Then $\overline{(\delta + \delta')}$ is a generator [**3**, **8**]. In this case, in order that $e^{t(\overline{\delta+\delta'})}$ has a ground state, it seems unnecessary that the δ-bound of δ' be smaller than or equal to one. If there is a sequence of finite type I subfactors with the identity of A whose union is a core for δ, then $e^{t(\overline{\delta+\delta'})}$ is approximately inner, and hence has a ground state [**11**]. Does $e^{t(\overline{\delta+\delta'})}$ have a ground state if α is approximately inner? We do not know even if δ' commutes with α.

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