

TWO EXACT SEQUENCES IN RATIONAL HOMOTOPY THEORY RELATING CUP PRODUCTS AND COMMUTATORS

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ABSTRACT. Let X be an $(n - 1)$ -connected topological space of finite rational type (i.e. $H_n(X; Q)$ is finite dimensional over Q for all n). Sullivan's notion of minimal model is used to derive two exact sequences involving the kernel of the cup product operation in dimension n and Whitehead products. The first of these generalizes both a theorem of John C. Wood [JCW] and a theorem of Dennis Sullivan [DS] and states that the kernel of the cup product map $H^1(X) \wedge H^1(X) \rightarrow H^2(X)$ is rationally the dual of the second factor of the lower central series of the fundamental group. Other examples are given in the last section.

1. Statement of the theorem. Let X be an $(n - 1)$ -connected topological space and let $H^n(X)$ denote its n th singular rational cohomology group. The object of this paper is to derive the following two exact sequences:

THEOREM 1. *The sequences*

$$(1.1) \quad 0 \rightarrow (\Gamma_2/\Gamma_3 \otimes Q)^* \xrightarrow{d} H^1 \wedge H^1 \xrightarrow{\cup} H^2 \quad \text{for } n = 1,$$

$$(1.2) \quad 0 \rightarrow H^{2n-1} \xrightarrow{-h} (\pi_{2n-1} \otimes Q)^* \xrightarrow{g} H^n \wedge H^n \xrightarrow{\cup} H^{2n} \quad \text{for } n > 1$$

are exact, where $\pi_i = \pi_i(X)$, and Γ_i is the i th term in the lower central series of π_1 , $H^i = H^i(X; Q)$, d is dual to the commutator map $[\]: \pi_1/\Gamma_2 \wedge \pi_1/\Gamma_2 \rightarrow \Gamma_2/\Gamma_3$, h is dual to the Hurewicz map \mathcal{H} , g is dual to the composition of $\mathcal{H} \otimes \mathcal{H}$ and the Whitehead product $[\]: \pi_n \otimes \pi_n \rightarrow \pi_{2n-1}$ and \wedge denotes exterior product if n is odd and symmetric tensor product if n is even.

The sequence (1.1) above is stated for the special case of compact oriented 3-manifolds in [DS] without proof. The case $\Gamma_2 = 0$ (i.e., π_1 abelian) of (1.1) gives Corollary 2 of [JCW]. Note that (1.1) applies to groups G and group cohomology if $X = K(G; 1)$ is used. The author is grateful to A. Libgober for inquiring about the validity of (1.1) in its stated generality. We note that versions of these exact sequences may also be derived from known spectral sequences [BK, LS] (see also

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[HS, 4.14; DQ]). The approach in this paper was chosen because of its elementary nature, assuming some properties of minimal models.

2. Notation and review. In this paper all coefficients will be in Q . A differential graded augmented algebra (DGA-alg) is a graded vector space $A = \bigoplus_{n \geq 0} A^n$ over Q with an associative, graded commutative multiplication $A^n \otimes A^m \rightarrow A^{n+m}$ and an augmentation $\varepsilon: A \rightarrow Q \rightarrow 0$ and a differential $d: A^n \rightarrow A^{n+1}$ which is a derivation of A [Mac]. All DGA-algs will be of finite Q type. Given a degree 0 cochain map $f: C \rightarrow D$ the “mapping cylinder” construction is the DGA-alg M_f with $(M_f)^n = C^n \oplus D^{n-1}$, $d(x, y) = (d(x), f(x) - d(y))$.

From the short exact sequence $0 \rightarrow D^{n-1} \rightarrow (M_f)^n \rightarrow C^n \rightarrow 0$, $i(y) = (0, -y)$, $q(x, y) = x$, we obtain the long exact sequence

$$(2.1) \quad \cdots \rightarrow H^{n-1}(C) \rightarrow H^{n-1}(D) \rightarrow H^n(f) \rightarrow H^n(C) \rightarrow \cdots,$$

where $H^*(f) = H^*(M_f)$. Note that the connecting homomorphism δ^* satisfies $\delta^* = -f^*$. If V is a graded vector space then $\Lambda(V)$ will denote the free graded commutative augmented algebra generated by V .

2.2. 1-minimal model. For a DGA-alg A such that $H^0(A) = Q$ there is an associated DGA-alg M_1 which possesses a filtration $M_{1,n} \leq M_{1,n+1}$, $n \geq 0$, with $M_{1,0} = Q$, $M_1 = \bigcup_{n \geq 0} M_{1,n}$, and a degree 0 cochain map $p: M_1 \rightarrow A$ that induces an isomorphism on H^1 and a monomorphism on H^2 . Inductively one has:

- (i) $V_1 = H^1(A)$ and $M_{1,1} = \Lambda(V_1)$ with trivial differential.
- (ii) $p_1: M_{1,1} \rightarrow A$ is given by a choice of algebra generators: $p_1(\text{cls}(a)) = a$.
- (iii) $M_{1,n+1} = M_{1,n} \otimes \Lambda(V_{n+1})$, where $V_{n+1} = H^2(p_n)$.
- (iv) The differential on $M_{1,n+1}$ is given by choosing a splitting of the natural quotient map $Z^2(p_n) \rightarrow H^2(p_n)$, say $s(v) = (m_v, a_v)$ and extending $d|_{M_{1,n}} = \text{old } d$, $d(v) = m_v$, $v \in V_{n+1}$, as a derivation.
- (v) $p_{n+1}: M_{1,n+1} \rightarrow A$ is induced by extending $p_{n+1}|_{M_{1,n}} = p_n$, $p_{n+1}(v) = a_v$, $v \in V_{n+1}$.

Note. Elements of V_n , $n \geq 1$, are to be considered as having degree 1. M_1 is called the 1-minimal model of A . It is unique up to isomorphism.

2.3. Minimal model. For a DGA-alg A such that $H^0(A) = Q$, and $H^1(A) = 0$ there is an associated DGA-alg M which possesses a filtration $M_n \leq M_{n+1}$, $n \geq 0$, with $M_1 = Q$, $M = \bigcup_{n \geq 0} M_n$, and a degree 0 cochain map $p: M \rightarrow A$ which induces an isomorphism in cohomology. Inductively one has:

- (i) $M_0 = M_1 = Q$, $V_2 = H^2(A)$ and $M_2 = \Lambda(V_2)$ with trivial differential.
- (ii) $p_2: M_2 \rightarrow A$ is given by a choice of algebra generators.
- (iii) $M_{n+1} = M_n \otimes \Lambda(V_{n+1})$, where $V_{n+1} = H^{n+2}(p_n)$.
- (iv) $d|_{M_n} = \text{old } d$, $d(v) = m_v$, where $s(v) = (m_v, a_v)$, as before.
- (v) $p_{n+1}: M_{n+1} \rightarrow A$ is induced by extending $p_{n+1}|_{M_n} = p_n$, $p_{n+1}(v) = a_v$.

Note. Elements of V_n , $n \geq 2$, are to be considered as having degree n . M is called the minimal model of A . It is unique up to isomorphism. The constructions above are in [GM].

2.4. REMARK. For a DGA-alg such that $H^0(A) = Q$ there is a notion of minimal model which combines the notions of 1-minimal model and minimal model [DS2, BG]. Using this one can obtain a “twisted version” of (1.1) and (1.2) combined for “nilpotent spaces”, i.e. spaces for which π_1 is nilpotent and acts nilpotently on π_n for all n .

3. Proof of Theorem 1. In this section let $A = A^*(X)$ be the rational de Rham complex of X .

3.1. Let V_n be as in 2.2 and $W_n = V_1 \oplus V_2 \oplus \cdots \oplus V_n$. We have that $W_n \rightarrow W_{n-1} \wedge W_{n-1} \hookrightarrow W_n \wedge W_n$ induces a Lie algebra structure: $d^*: W_n^* \wedge W_n^* \rightarrow W_n^*$. The main theorem on 1-minimal models states that $W_n^* = \mathcal{L}(\pi_1(X)/\Gamma_{n+1})$ (the Mal'cev Lie algebra of the nilpotent group π_1/Γ_{n+1}). Thus we have $\mathcal{L}(\pi_1/\Gamma_3) = W_2^* = V_1^* \oplus V_2^*$ additively.

The nontrivial part of the Lie bracket map is $[\cdot, \cdot]: V_1^* \wedge V_1^* \rightarrow V_2^*$ and the nilpotency class of π_1/Γ_3 is ≤ 2 so its Mal'cev Lie algebra is the same as the graded Lie algebra associated to its lower central series (see [LP, 7] for an example of a nilpotent group such that $\mathcal{L}(G) \neq \text{gr}(G)$). Since $V_1^* = (\pi_1/\Gamma_2 \otimes Q)^*$ it follows that $V_2^* = (\Gamma_2/\Gamma_3 \otimes Q)^*$ and that $[\cdot, \cdot]$ is given by group commutator. Now in this case the long exact sequence of the mapping cylinder (2.1) becomes

$$(3.2) \quad \begin{array}{ccccccc} 0 & \rightarrow & H^2(p_1) & \rightarrow & H^2(M_{1,1}) & \rightarrow & H^2(A) \rightarrow \cdots \\ & & \downarrow & & \downarrow & & \\ & & V_2 & \rightarrow & \Lambda^2(H^1) & & \end{array}$$

and the map d^* is easily identified as $[\cdot, \cdot]$. With these identifications the exact sequence (1.1) is just (3.2).

3.3. Let V_n be as in 2.3 and $V_* = \bigoplus_{n \geq 2} V_n$ and let $M^+ = \bigoplus_{j > 0} M^j$. The main theorem on minimal models states that for a 1-connected space X , $V_n^* = \pi_n$ and the differential from 2.2(iv) induces a map

$$\begin{array}{ccc} M^+/(M^+ \cdot M^+) & \rightarrow & (M^+ \cdot M^+)/ (M^+ \cdot M^+ \cdot M^+) \\ \downarrow & & \downarrow \\ d: V_* & \rightarrow & V_* \wedge V_* \end{array}$$

where d is dual to the Whitehead product $[\cdot, \cdot]: \pi_* \wedge \pi_* \rightarrow \pi_*$ (a degree -1 map) [GM, AA, BG, DS2]. To obtain the exact sequence (1.2) we assume that X is $(n-1)$ -connected, $n > 1$. We have

3.4. LEMMA. (i) For $1 \leq k \leq n-2$

$$\begin{array}{ccccccc} H^{n+k}(M_{n+k-1}) & \rightarrow & H^{n+k}(A) & \rightarrow & H^{n+k+1}(p_{n+k-1}) & \rightarrow & H^{n+k+1}(M_{n+k-1}) \\ \parallel & & \downarrow & & \downarrow & & \parallel \\ 0 & \rightarrow & H^{n+k} & \rightarrow & V_{n+k} & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ & & (H_{n+k})^* & \rightarrow & (\pi_{n+k})^* & & \\ & & \cong & & & & \end{array}$$

where the bottom arrow is dual to the Hurewicz map.

(ii) For $k = n - 1$

$$\begin{array}{ccccccccc}
 H^{2n-1}(M_{2n-2}) & \rightarrow & H^{2n-1}(A) & \rightarrow & H^{2n}(p_{2n-2}) & \rightarrow & H^{2n}(M_{2n-2}) & \rightarrow & H^{2n}(A) \\
 \parallel & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 & & H^{2n-1} & & V_{2n-1} & & H^n \otimes H^n & & H^{2n} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & (H_{2n-1})^* & & (\pi_{2n-1})^* & & (H_n \otimes H_n)^* & & \\
 & & & & & & \searrow \tau & & \downarrow \\
 & & & & & & & & (\pi_n \otimes \pi_n)^*
 \end{array}$$

where the diagonal map is dual to the Whitehead product.

PROOF. Using the minimal model construction 2.3 we have $M_j = Q$, $j < n$, and $M_n = \Lambda(V_n) = \Lambda(H^n)$ with $d = 0$. Thus (2.1) becomes:

$$\begin{array}{ccccccccc}
 \rightarrow H^{n+1}(M_n) & \rightarrow & H^{n+1}(A) & \rightarrow & H^{n+2}(p_n) & \rightarrow & H^{n+2}(M_n) & \rightarrow & \\
 \parallel & & \downarrow & & \downarrow & & & & \\
 0 & & H^{n+1}(X) & & V_{n+1} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & (H_{n+1}(X))^* & \rightarrow & (\pi_{n+1}(X))^* & & & &
 \end{array}$$

By the general theory, the vertical maps may be chosen so that the bottom map is dual to the Hurewicz map [GM]. By definition, $M_{n+1} = M_n \otimes \Lambda(V_{n+1})$ and in this case we can choose a splitting $s: H^{n+2}(p_n) \rightarrow Z^{n+2}(p_n)$ such that $s(v) = (0, a_v)$ to get the differential. Thus $d(v) = 0$ and $p_{n+1}(v) = a_v$. This is the first step in an induction which shows that for all j such that $1 \leq j \leq n - 2$, $M_{n+j} = \Lambda(V_n \oplus V_{n+1} \oplus \cdots \oplus V_{n+j})$ with trivial differential. This statement is equivalent to (i) (choosing the vertical isomorphisms correctly). It is now clear that (ii) is just the exact sequence (2.1) with the vertical maps coming from the general theory. For the diagonal map τ note that $q^*(v) = \text{cls}(m_v)$ where we have chosen a splitting $s: H^{2n}(p_{2n-2}) \rightarrow Z^{2n}(p_{2n-2})$, $s(v) = (m_v, a_v)$. Since $m_v \in Z^{2n}(M_{2n-2}) = \Lambda(V_n \oplus \cdots \oplus V_{2n-2})$, it can at most be quadratic; but we also have that $d(v) = m_v$. Since the Whitehead product is determined by the quadratic part of the differential [DS, AA] we are done.

4. Examples. In [DS] a special case of (1.1) is used to show that $S^1 \times S^1 \times S^1$ is not the link of an isolated singularity of a complex algebraic surface. It is noted in [JCW] that the case of π_1 abelian yields a result of H. Hopf: If $\pi_1(X)$ is Abelian, then the first and second Betti numbers of X satisfy $\beta_1(\beta_1 - 1)/2 \leq \beta_2$ [HH].

In the special case that $X = G/D$ is a nilmanifold, i.e. the homogeneous space determined by a nilpotent Lie group G and a discrete uniform subgroup D , (1.1) follows immediately from the well-known fact that if \mathcal{L} is the rational graded Lie algebra associated to the lower central series of D , then the 1-minimal model of X is just the Koszul complex of \mathcal{L} [DS2, LP]. In this context, (1.1) leads to an explicit construction of Massey products [JPM] in terms of the commutator structure.

4.1. *Higher dimensions.* In certain cases the present methods can be used to derive a generalization of the higher dimensional statements in [JCW, Theorem 1] by giving the kernel of the cup product in an explicit form. (1.2) is an example of this.

Note that (1.2) also contains the presumably well-known result that the $(2n - 1)$ st homotopy group of an $(n - 1)$ -connected $2n$ -manifold is the kernel of cup product in dimension n rationally.

4.2. *Final remarks.* It should be noted that there are spaces of interest for which the theory fails. For example let V^n be the algebraic variety defined by the equation $\sum_{i=1}^{n+1} (f_{q_i}) = P_i = 0$, where $q_1 = \prod_{i=1}^{n+1} p_i/p_i$, $1 \leq i \leq n + 1$, and f_{q_i} denotes a generic form of degree q_i . It is shown in [AL] that $\pi_n(CP_{n+1} - (V^n \cup H)) \otimes Q \neq 0$, where H is a generic hyperplane. For example, the space corresponding to the equation $(f_{21})^2 + (f_{14})^3 + (f_6)^7 = 0$ has $\pi_1 = Z$, $H_2 = 0$. If the fundamental group acted nilpotently on higher homotopy the twisted theory mentioned in 2.4 would give that $\pi_2 = 0$.

Finally, the particular construction of the minimal model used in this paper was chosen to accomodate our special purpose. For other purposes other forms of the minimal model may be more useful. This is well illustrated in [HS].

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