Abstract. Let $X$ be an analytic space (e.g., a complete metric space). We prove that any point-countable $\mathcal{F}_\sigma$-set cover of $X$ either has $\sigma$-discrete refinement, or else there is a compact subset of $X$ not covered by any countable subfamily of the cover. It follows that any point-countable $\mathcal{F}_\sigma$-additive family in $X$ has a $\sigma$-discrete refinement. This is used to show that any weakly $F_\sigma$-measurable multimap, defined on $X$ and taking nonempty, closed and separable values in a complete metric space, has a selector of the first Baire class.

1. Introduction. A classical theorem due to Souslin states that any analytic subset of a complete separable metric space is either countable or contains an uncountable compact set (and thus a copy of the Cantor set) [11, §32]. For nonseparable complete metric spaces, this theorem remains true if we replace “countable” by “$\sigma$-discrete” (i.e., a countable union of closed discrete subsets). This was first shown by A. H. Stone [18] for Borel subsets, and then by A. G. El’kin [1] for analytic sets. An immediate corollary is that any subset of a complete metric space, all of whose subsets are analytic, must be $\sigma$-discrete.

A useful “set version” of this last result was obtained in [6]: Any disjoint family of subsets of a complete metric space $X$ with the property that the union of every subfamily is analytic in $X$ has a $\sigma$-discrete refinement\(^1\) (equivalently, is $\sigma$-discretely decomposable\(^2\)). This result was subsequently generalized to point-finite families [13] and to certain nonmetrizable spaces [5]. (See also [2 and 3] for consistency results along this line.) It is natural to ask whether there is a set version of El’kin’s result: Does each partition $\mathcal{E}$ of a complete metric space into analytic subsets have the property that either $\mathcal{E}$ has a $\sigma$-discrete refinement, or there is a compact set meeting uncountably many members of $\mathcal{E}$? The answer turns out to be negative even for $G_\delta$-set partitions (see [14 and 18, §5]). However, G. Koumoullis [14] has recently obtained the following interesting result.
Theorem 1.1 [14]. Let $\mathcal{E}$ be an $F_\sigma$-set partition of an analytic space $X$. Then either $\mathcal{E}$ is $\sigma$-discretely decomposable, or $X$ contains a compact set meeting uncountably many members of $\mathcal{E}$.

The primary purpose of this note is to give the following extension of Theorem 1.1.

Theorem 2.1. Let $\mathcal{E}$ be a point-countable $F_\sigma$-set cover of an analytic space $X$. Then, either $\mathcal{E}$ has a $\sigma$-discrete refinement, or $X$ contains a compact subset that is not covered by any countable subfamily of $\mathcal{E}$.

Theorem 2.1 implies Theorem 1.1, since a disjoint family having a $\sigma$-discrete refinement is easily seen to be $\sigma$-discretely decomposable. Note also that the two possible properties of $\mathcal{E}$ in Theorem 2.1 are mutually exclusive: For if $\mathcal{E}$ has a $\sigma$-discrete refinement and $C$ is any compact subset of $X$, then $C$ meets at most countably many members of the refinement, and so is covered by some countable subfamily of $\mathcal{E}$. Further, we note that Theorem 2.1 cannot be sharpened by replacing “$\sigma$-discrete refinement” with “$\sigma$-discretely decomposable”: Hausdorff [12] has shown that $\mathbb{R}$ is the union of a strictly decreasing sequence of $F_\sigma$-sets indexed by the countable ordinals. Such a family is point-countable but could not be $\sigma$-discretely decomposable, since this would imply the existence of an uncountable discrete subset of $\mathbb{R}$. That point-countability in Theorem 2.1 cannot be omitted follows from [17, Example 1.4].

If $\mathcal{L}$ is any collection of sets, we say that a family of sets is $\mathcal{L}$-additive if the union of each subfamily belongs to $\mathcal{L}$. In §3 we prove the following consequence of Theorem 2.1.

Theorem 3.3. Let $X$ be an analytic space. Then every point-countable $F_\sigma$-additive family of subsets of $X$ has a $\sigma$-discrete refinement.

In contrast to the situation with Theorem 2.1, Theorem 3.3 is believed to hold for Borel sets of arbitrary class, although no proof is known even for $G_\delta$-additive families. R. Pol [17] has shown that a point-countable (extended) Borel-additive family in an analytic space has a $\sigma$-discrete refinement, provided the members of the family have weight at most $\aleph_1$. Theorem 3.3 lends support to the conjecture that the weight restriction can be omitted. Under additional axioms of set theory this holds true even for arbitrary metric spaces (see [3, §4]).

We conclude with an application of Theorem 3.3 to the study of measurable selections by proving the following.

Theorem 4.1. Let $T$ be a regular analytic space, $X$ a metric space, and $F: T \to X$ a multimap having nonempty, separable and $\rho$-complete values, where $\rho$ is a metric for

$3$ Analytic spaces are defined in §2.

$4$ A collection is point-countable if no point belongs to more than countably many members of the collection.
X. Assume that $F$ is weakly $F_\sigma$-measurable; i.e., for each open $U$ in $X$,

$$F^{-1}(U) = \{ t \in T: F(t) \cap U \neq \emptyset \}$$

is an $F_\sigma$-set of $T$. Then $F$ has a selector of the first class (i.e., there is a map $f: T \to X$ such that $f(t) \in F(t)$, for all $t \in T$, and $f^{-1}(U)$ is an $F_\sigma$-set of $T$ for all open sets $U$ of $X$).

2. Analytic spaces and the proof of Theorem 2.1. By an analytic space we mean any Hausdorff space $X$ that is a continuous, base-$\sigma$-discrete image of a complete metric space. A map $f: Z \to X$ is base-$\sigma$-discrete [16] if to each discrete family $\mathcal{A}$ in $Z$ there corresponds a $\sigma$-discrete family $\mathcal{B}$ in $X$ such that $f(A)$ is the union of some subfamily of $\mathcal{B}$ for each $A$ in $\mathcal{A}$ ($\mathcal{B}$ is said to be a base for $\{ f(A): A \in \mathcal{A} \}$). It is easy to see that any (Hausdorff) continuous image of a complete separable metric space is an analytic space (discrete families are countable). Also, any analytic subset of a complete metric space is an analytic space [9].

**Proof of Theorem 2.1.** We first consider the case when $X$ has a complete metric $d$. Let $F$ denote the largest closed subset of $X$ such that no nonempty open subset of $F$ is covered by countably many sets from $\mathcal{E}$. The existence of $F$ follows from [19, Theorem 1] where $F$ is called the “non-locally-$P$ kernel of $X$”, $P$ being the collection of all subsets of $X$ which are covered by countably many sets from $\mathcal{E}$. If $F = \emptyset$, then by [19, Theorem 4'] $X$ is the union of a $\sigma$-discrete family of closed sets each of which is covered by countably many sets from $\mathcal{E}$. Clearly, this yields a $\sigma$-discrete refinement of $\mathcal{E}$. Assuming $F \neq \emptyset$, we now construct a compact subset of $X$ that is not covered by any countable subfamily of $\mathcal{E}$.

Let $S$ denote the set of all finite sequences of natural numbers (including $\emptyset$), and define $\|s\| = 0$, and

$$\|s\| = \sum_{i=1}^{n} s_i, \quad \text{where} \ s = (s_1, \ldots, s_n) \in S.$$  

For $s \in S$, $l(s)$ denotes the length of $s$. For each $s \in S$ we now define by induction on $l(s)$ points $x_s \in F$ and sets $E_s \in \mathcal{E}$ satisfying the following:

(i) $x_s \in E_s$;

(ii) if $l(r) < l(s)$ and $x_r \in E \in \mathcal{E}$, then $x_s \not\in E$;

(iii) if $s = (t, n)$ for some $t \in S$, then $d(x_{nt}, x_s) < 1/2^\|s\|$.

Choose $x_\emptyset \in F$ arbitrarily. Suppose $x_s$ and $E_s$ are known for all $s \in S$ with $l(s) \leq m$ for some $m \geq 0$. Given $s = (t, n)$ of length $m + 1$, let $B$ be the basic neighborhood about $x_t$ of $d$-radius $1/2^\|s\|$, and let

$$\mathcal{E}_s = \{ E \in \mathcal{E}: x_s \in E \text{ for some } r \text{ with } l(r) < l(s) \}.$$  

Since $\mathcal{E}_s$ is countable and $x_s \in F$, $B \cap F - \bigcup \mathcal{E}_s$ is not empty, so there is some $E_s \in \mathcal{E} - \mathcal{E}_s$ and some point $x_s \in B \cap F \cap E_s$. Properties (i)-(iii) are clearly satisfied by $x_s$ and $E_s$.
We let $Q = \{ x_s : s \in S \}$. Property (iii) above ensures that $Q$ is $d$-totally bounded, and so $K = \overline{\mathcal{X} Q}$ is compact. Now, for each $E \in \mathcal{E}$, the interior of $E \cap K$ relative to $K$ must be empty; otherwise, by the denseness of $Q$ and since $x_{(s,n)}$ converges to $x_s$, we would have both $x_s$ and $x_{(s,n)}$ belonging to $E$, for some $s$ and $n$, in contradiction with (ii) above. Since $E \cap K$ is also an $F_\sigma$-set in $K$, this implies that each $E \cap K$ is of first category in $K$. Since $K$ is a Baire space, it follows that $K$ cannot be covered by countably many sets from $\mathcal{E}$. This completes the proof in the case when $X$ is completely metrizable.

If $X$ is analytic, then we can find a complete metric space $Z$ and a continuous surjection $f : Z \to X$ with the property that the image of any $\sigma$-discrete family in $Z$ has a $\sigma$-discrete base (and hence refinement) relative to $X$. Now $\{ f^{-1}(E) : E \in \mathcal{E} \}$ is a point-countable cover of $Z$ by $F_\sigma$-sets, and so must either have a $\sigma$-discrete refinement, or else $Z$ contains a compact set $C$ not covered by countably many sets of the form $f^{-1}(E)$, $E \in \mathcal{E}$. But then, by the properties of $f$, $\mathcal{E}$ either has a $\sigma$-discrete refinement, or the compact set $f(C)$ exists and is not covered by any countable subfamily of $\mathcal{E}$. □

We remark that the above proof makes use of several techniques suggested by [14], some of which G. Koumoullis attributes to D. H. Fremlin. □

3. Weakly discrete and extended Borel-additive families. Let $\mathcal{E}$ be a family of subsets of $X$. Following R. Pol [17] we say that $A \subseteq X$ is $\mathcal{E}$-discrete provided, for each $a \in A$, there is an $E_a \in \mathcal{E}$ satisfying $E_a \cap A = \{a\}$; if $X$ is a topological space, we say that $\mathcal{E}$ is weakly discrete if every $\mathcal{E}$-discrete set is a $\sigma$-discrete set in $X$. Our interest in weakly discrete families stems from the following.

3.1 Lemma. If $\mathcal{E}$ is a point-countable weakly discrete family of subsets of a space $X$ and $L$ is a Lindelöf subspace of $X$, then $L \cap (\bigcup \mathcal{E})$ is covered by a countable subfamily of $\mathcal{E}$.

Proof. Suppose $L \cap (\bigcup \mathcal{E})$ is not covered by any countable subfamily of $\mathcal{E}$. Then, by induction over the countable ordinals, we can easily define a set $A = \{ x_\alpha : \alpha < \omega_1 \}$ contained in $L$ such that, for all $\beta < \alpha$, if $x_\beta \in E \in \mathcal{E}$, then $x_\alpha \notin E$. Then for any $E_\alpha \in \mathcal{E}$ with $x_\alpha \in E_\alpha$, we have $A \cap E_\alpha = \{ x_\alpha \}$. Thus $A$ is $\mathcal{E}$-discrete, and so $A$ can be written as a countable union of closed discrete subsets of $X$. But this implies that $L$ has an uncountable closed discrete subset, contradicting the fact that $L$ is Lindelöf. □

By the extended Borel sets of a topological space $X$ we mean the smallest $\sigma$-algebra of subsets of $X$ which contains the open sets and is closed to the operation of discrete union [9]. By a Souslin set of $X$ we mean, as usual, a subset of $X$ obtained by applying the Souslin operation to the closed sets of $X$. For an analytic space $X$ it can be shown that the extended Borel sets coincide with the family of all subsets $A$ of $X$ such that $A$ and $X - A$ are Souslin sets of $X$ (see, e.g., [10]).

I would like to thank Professor G. Koumoullis for providing a preprint of [14].
We now prove a slight refinement of a result due to R. Pol in the metrizable case \[17\], although the proof given here is considerably less technical.

**Theorem 3.2.** Let \( X \) be an analytic space, and let \( \mathcal{E} \) be an extended Borel-additive family of subsets of \( X \). Then \( \mathcal{E} \) is weakly discrete.

**Proof.** First assume that \( X \) is a complete metric space. Let \( A \subset X \) be such that, for each \( a \in A \), \( A \cap E_a = \{a\} \) for some \( E_a \in \mathcal{E} \). For any nonempty subset \( B \) of \( A \), we have

\[
B = A \cap \left( \bigcup_{b \in B} E_b \right),
\]

so \( B \) is a Souslin set relative to \( A \). Thus, if we can show that \( A \) is a Souslin set in \( X \), then \( A \) will be an analytic metric space all of whose subsets are analytic, and hence a \( \sigma \)-discrete set by the theorem of El'kin. Since

\[
A = \bigcup_{a \in A} E_a - \bigcup_{a \in A} E_a \cap (X - \{a\}),
\]

we need only show that the set \( C = \bigcup_{a \in A} E_a \cap (X - \{a\}) \) is extended Borel in \( X \). Since \( X \) is metrizable, let \( \bigcup_{n \in \mathbb{N}} \mathcal{B}_n \) be an open base for \( X \) with each \( \mathcal{B}_n \) a discrete family in \( X \). For each \( B \in \bigcup_{n \in \mathbb{N}} \mathcal{B}_n \) define

\[
E_B = \bigcup \{ E_a : a \in A \text{ and } B \subset X - \{a\} \},
\]

and note that

\[
C = \bigcup_n \bigcup_{B \in \mathcal{B}_n} E_B \cap B.
\]

Since \( \{ E_B \cap B : B \in \mathcal{B}_n \} \) is a discrete family of extended Borel sets for each \( n \), it follows that \( C \) is extended Borel. This proves that \( \mathcal{E} \) is weakly discrete when \( X \) is completely metrizable.

For \( X \) an analytic space, let \( Z \) be a complete metric space, and let \( f : Z \to X \) be a continuous, base-\( \sigma \)-discrete surjection. It is clear that

\[
f^{-1}(\mathcal{E}) = \{ f^{-1}(E) : E \in \mathcal{E} \}
\]

is an extended Borel-additive family in \( Z \). Now let \( A \subset X \) and \( E_a \) be as before, and let \( Z_A = \{ z_a : a \in A \} \) be such that \( z_a \in f^{-1}(a) \) for each \( a \in A \). Then \( Z_A \) is \( f^{-1}(\mathcal{E}) \)-discrete, and thus a \( \sigma \)-discrete set by the above. But then \( A = f(Z_A) \) is \( \sigma \)-discrete, since \( f \) is a base-\( \sigma \)-discrete map.

**Proof of Theorem 3.3.** Let \( \mathcal{E} \) be a point-countable, \( F_\alpha \)-additive family of subsets of the analytic space \( X \), and let \( Y = \bigcup \mathcal{E} \). Then \( Y \) is an analytic space, and \( \mathcal{E} \) is a weakly discrete family in \( X \) by Theorem 3.2. In view of Lemma 3.1 and Theorem 2.1, it follows that \( \mathcal{E} \) must have a refinement \( \mathcal{R} \) that is \( \sigma \)-discrete relative to \( Y \). Since \( Y \) is an \( F_\alpha \)-set in \( X \), \( \mathcal{R} \) is easily seen to have a \( \sigma \)-discrete refinement relative to \( X \).

\[\square\]

**4. Proof of Theorem 4.1.** For \( n = 1, 2, \ldots \), let \( \mathcal{U}_n \) be a locally finite cover of \( X \) by open sets having \( \rho \)-diameter \( < 1/n \). Since each separable subset of \( X \) can meet at most countably many members of a locally finite family, our assumptions imply that
\{ F^{-1}(U) : U \in \mathcal{U}_1 \} is a point-countable $F_\sigma$-additive cover of $T$, and so has a $\sigma$-discrete refinement $\mathcal{M}$. We may assume that $\mathcal{M}$ is the union of $\mathcal{M}_m (m = 1, 2, \ldots)$, where each $\mathcal{M}_m$ is a discrete family of $F_\sigma$-sets in $T$. Then $\mathcal{M}_m = \bigcup \mathcal{M}_m$ is an $F_\sigma$-set of $T$ for each $m$, and, applying the countable reduction principle [15, p. 350], there exists a sequence $\{ H_m \}_{m \geq 1}$ of pairwise disjoint $F_\sigma$-sets such that

$$H_m \subset M_m \quad \text{and} \quad T = \bigcup_{m=1}^{\infty} H_m.$$  

(Here we have used the fact that in a regular analytic space, open sets are $F_\sigma$-sets, so the above reduction property is valid.) It follows that

$$\mathcal{M} = \{ M \cap H_m : M \in \mathcal{M}_m, m = 1, 2, \ldots \}$$

is a disjoint, $\sigma$-discrete, $F_\sigma$-additive refinement of $\{ F^{-1}(U) : U \in \mathcal{U}_1 \}$. For each $H \in \mathcal{M}$ choose some $U_H \in \mathcal{U}_1$ such that $H \subset F^{-1}(U_H)$, and let $H(U) = \bigcup \{ H : U_H = U \}$ for each $U \in \mathcal{U}_1$. Then the family $\{ H(U) : U \in \mathcal{U}_1 \}$ is also a disjoint, $\sigma$-discrete, $F_\sigma$-additive cover of $T$ and $H(U) \subset F^{-1}(U)$ for each $U \in \mathcal{U}_1$. Now define the multimap $F_1 : T \to X$ by

$$F_1(t) = F(t) \cap U \quad \text{iff} \quad t \in H(U), \quad U \in \mathcal{U}_1.$$  

For any open $V \subset X$, one has

$$F^{-1}_1(V) = \bigcup \{ F^{-1}(U \cap V) \cap H(U) : U \in \mathcal{U}_1 \},$$

and this is an $F_\sigma$-set of $T$ as the union of a $\sigma$-discrete collection of $F_\sigma$-sets. Thus $F_1$ is weakly $F_\sigma$-measurable and has nonempty separable values. It follows that $\{ F_1^{-1}(U) : U \in \mathcal{U}_2 \}$ is a point-countable $F_\sigma$-additive cover of $T$, and we may apply the above argument again to obtain a family $\{ K(U) : U \in \mathcal{U}_2 \}$ that is disjoint, $\sigma$-discrete, $F_\sigma$-additive, covers $T$, and is such that $K(U) \subset F_1^{-1}(U)$ for each $U \in \mathcal{U}_2$. We proceed to define $F_2 : T \to X$ by

$$F_2(t) = F_1(t) \cap U \quad \text{iff} \quad t \in K(U), \quad U \in \mathcal{U}_2,$$

and observe as before that $F_2$ is weakly $F_\sigma$-measurable and has nonempty separable values. In this way we generate a sequence of weakly $F_\sigma$-measurable multimaps $F_n : T \to X$ satisfying $F(t) \supset F_1(t) \supset \cdots \supset F_n(t) \supset \cdots$, and $F_n(t)$ is nonempty and has $\rho$-diameter $< 1/n$.

By the $\rho$-completeness of the values of $F$, we can define a map $f : T \to X$ by taking $f(t)$ to be the unique member of $\bigcap_n F_n(t)$. Now, for any open $U \subset X$, we have

$$f^{-1}(U) = \bigcup_{n=1}^{\infty} F_n^{-1}(U_n),$$

where $U_n = \{ x \in U : \rho\text{-dist}(x, X - U) > 1/n \}$. Thus $f^{-1}(U)$ is an $F_\sigma$-set of $T$, proving $f$ is a selector for $F$ of the first class. \qed

Remark. For recent measurable selection theorems along lines similar to the above, see [4 and 8].

We conclude with an example which shows the assumption that $F$ has separable values in Theorem 4.1 cannot be omitted, even when $T$ is completely metrizable.
Example. There exists a weakly $F_\sigma$-measurable multimap $F$ from the Baire space $B(\omega_1)$ to the space $\omega_1$ with the discrete topology, having no extended Borel measurable selector.

Proof. Let $B(\omega_1) = \omega_1^\mathbb{N}$ with the product topology, and define, for each $\alpha < \omega_1$,

$$S_\alpha = \{ x \in B(\omega_1) : x(n) \leq \alpha \text{ for all } n \in \mathbb{N} \}.$$

It is easy to check that $\{ S_\alpha : \alpha < \omega_1 \}$ is an increasing, $F_\sigma$-additive cover of $B(\omega_1)$ by closed, separable subsets. Thus, defining $F : B(\omega_1) \to \omega_1$ by

$$F(x) = \{ \alpha < \omega_1 : x \in S_\alpha \},$$

we have $F^{-1}(\alpha) = S_\alpha$, for each $\alpha$, and so $F$ is weakly $F_\sigma$-measurable. Now, if $f$ were an extended Borel measurable selector for $F$, then $\{ f^{-1}(\alpha) : \alpha < \omega_1 \}$ would be a disjoint extended Borel-additive family in $B(\omega_1)$, and so $\sigma$-discrete by [6, Theorem 2]. Since this family refines $\{ S_\alpha : \alpha < \omega_1 \}$, it would follow that the latter has a $\sigma$-discrete refinement and, hence, that $B(\omega_1)$ can be covered by a $\sigma$-discrete collection of separable subsets. But this would imply that $B(\omega_1)$ is $\sigma$-locally of weight $< \omega_1$, in contradiction to a theorem of A. H. Stone [20, 2.1(7)]. □

References


Department of Mathematics, University of Connecticut, Storrs, Connecticut 06268