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F_σ-SET COVERS OF ANALYTIC SPACES AND FIRST CLASS SELECTORS

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ABSTRACT. Let X be an analytic space (e.g., a complete metric space). We prove that any point-countable F_σ-set cover of X either has σ-discrete refinement, or else there is a compact subset of X not covered by any countable subfamily of the cover. It follows that any point-countable F_σ-additive family in X has a σ-discrete refinement. This is used to show that any weakly F_σ-measurable multimap, defined on X and taking nonempty, closed and separable values in a complete metric space, has a selector of the first Baire class.

1. Introduction. A classical theorem due to Souslin states that any analytic subset of a complete separable metric space is either countable or contains an uncountable compact set (and thus a copy of the Cantor set) [**11**, §32]. For nonseparable complete metric spaces, this theorem remains true if we replace "countable" by "σ-discrete" (i.e., a countable union of closed discrete subsets). This was first shown by A. H. Stone [**18**] for Borel subsets, and then by A. G. El'kin [**1**] for analytic sets. An immediate corollary is that any subset of a complete metric space, all of whose subsets are analytic, must be σ-discrete.

A useful "set version" of this last result was obtained in [**6**]: Any disjoint family of subsets of a complete metric space X with the property that the union of every subfamily is analytic in X has a σ-discrete refinement[1] (equivalently, is σ-discretely decomposable[2]). This result was subsequently generalized to point-finite families [**13**] and to certain nonmetrizable spaces [**5**]. (See also [**2** and **3**] for consistency results along this line.) It is natural to ask whether there is a set version of El'kin's result: Does each partition $\mathscr{E}$ of a complete metric space into analytic subsets have the property that either $\mathscr{E}$ has a σ-discrete refinement, or there is a compact set meeting uncountably many members of $\mathscr{E}$? The answer turns out to be negative even for G_δ-set partitions (see [**14** and **18**, §5]). However, G. Koumoullis [**14**] has recently obtained the following interesting result.

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[1] It is important to note that any family of subsets of a space X which is discrete relative to its union has a σ-discrete refinement relative to the space X, provided that all open subsets of X are F_σ-sets or X has a σ-discrete network [7, Lemma 2.2].

[2] $\{E_a\colon a \in A\}$ is *σ-discretely decomposable* if, for each $a \in A$, $E_a = E_{a,n}$ $(n \in \mathbf{N})$ and $\{E_{a,n}\colon a \in A\}$ is discrete for each n.

THEOREM 1.1 [**14**]. *Let $\mathscr{E}$ be an F_σ-set partition of an analytic space X.*[3] *Then either $\mathscr{E}$ is σ-discretely decomposable, or X contains a compact set meeting uncountably many members of $\mathscr{E}$.*

The primary purpose of this note is to give the following extension of Theorem 1.1.

THEOREM 2.1. *Let $\mathscr{E}$ be a point-countable*[4] *F_σ-set cover of an analytic space X. Then, either $\mathscr{E}$ has a σ-discrete refinement, or X contains a compact subset that is not covered by any countable subfamily of $\mathscr{E}$.*

Theorem 2.1 implies Theorem 1.1, since a disjoint family having a σ-discrete refinement is easily seen to be σ-discretely decomposable. Note also that the two possible properties of $\mathscr{E}$ in Theorem 2.1 are mutually exclusive: For if $\mathscr{E}$ has a σ-discrete refinement and C is any compact subset of X, then C meets at most countably many members of the refinement, and so is covered by some countable subfamily of $\mathscr{E}$. Further, we note that Theorem 2.1 cannot be sharpened by replacing "σ-discrete refinement" with "σ-discretely decomposable": Hausdorff [**12**] has shown that **R** is the union of a strictly decreasing sequence of F_σ-sets indexed by the countable ordinals. Such a family is point-countable but could not be σ-discretely decomposable, since this would imply the existence of an uncountable discrete subset of **R**. That point-countability in Theorem 2.1 cannot be omitted follows from [**17**, Example 1.4].

If $\mathscr{L}$ is any collection of sets, we say that a family of sets is $\mathscr{L}$-additive if the union of each subfamily belongs to $\mathscr{L}$. In §3 we prove the following consequence of Theorem 2.1.

THEOREM 3.3. *Let X be an analytic space. Then every point-countable F_σ-additive family of subsets of X has a σ-discrete refinement.*

In contrast to the situation with Theorem 2.1, Theorem 3.3 is believed to hold for Borel sets of arbitrary class, although no proof is known even for G_δ-additive families. R. Pol [**17**] has shown that a point-countable (extended) Borel-additive family in an analytic space has a σ-discrete refinement, provided the members of the family have weight at most $\aleph_1$. Theorem 3.3 lends support to the conjecture that the weight restriction can be omitted. Under additional axioms of set theory this holds true even for arbitrary metric spaces (see [**3**, §4]).

We conclude with an application of Theorem 3.3 to the study of measurable selections by proving the following.

THEOREM 4.1. *Let T be a regular analytic space, X a metric space, and $F\colon T \to X$ a multimap having nonempty, separable and ρ-complete values, where ρ is a metric for*

[3]Analytic spaces are defined in §2.

[4]A collection is *point-countable* if no point belongs to more than countably many members of the collection.

X. Assume that F is weakly F_σ-measurable; i.e., for each open U in X,

$$F^{-1}(U) = \{t \in T\colon F(t) \cap U \neq \varnothing\}$$

is an F_σ-set of T. Then F has a selector of the first class (i.e., there is a map $f\colon T \to X$ such that $f(t) \in F(t)$, for all $t \in T$, and $f^{-1}(U)$ is an F_σ-set of T for all open sets U of X).

2. Analytic spaces and the proof of Theorem 2.1. By an *analytic space* we mean any Hausdorff space X that is a continuous, base-σ-discrete image of a complete metric space. A map $f\colon Z \to X$ is *base-σ-discrete* [**16**] if to each discrete family $\mathscr{A}$ in Z there corresponds a σ-discrete family $\mathscr{B}$ in X such that $f(A)$ is the union of some subfamily of $\mathscr{B}$ for each A in $\mathscr{A}$ ($\mathscr{B}$ is said to be a *base* for $\{f(A)\colon A \in \mathscr{A}\}$). It is easy to see that any (Hausdorff) continuous image of a complete separable metric space is an analytic space (discrete families are countable). Also, any analytic subset of a complete metric space is an analytic space [**9**].

PROOF OF THEOREM 2.1. We first consider the case when X has a complete metric d. Let F denote the largest closed subset of X such that no nonempty open subset of F is covered by countably many sets from $\mathscr{E}$. The existence of F follows from [**19**, Theorem 1] where F is called the "non-locally-P kernel of X", P being the collection of all subsets of X which are covered by countably many sets from $\mathscr{E}$. If $F = \varnothing$, then by [**19**, Theorem 4′] X is the union of a σ-discrete family of closed sets each of which is covered by countably many sets from $\mathscr{E}$. Clearly, this yields a σ-discrete refinement of $\mathscr{E}$. Assuming $F \neq \varnothing$, we now construct a compact subset of X that is not covered by any countable subfamily of $\mathscr{E}$.

Let S denote the set of all finite sequences of natural numbers (including $\varnothing$), and define $\|\varnothing\| = 0$, and

$$\|s\| = \sum_{i=1}^{n} s_i, \quad \text{where } s = (s_1, \ldots, s_n) \in S.$$

For $s \in S$, $l(s)$ denotes the length of s. For each $s \in S$ we now define by induction on $l(s)$ points $x_s \in F$ and sets $E_s \in \mathscr{E}$ satisfying the following:

(i) $x_s \in E_s$;

(ii) if $l(r) < l(s)$ and $x_r \in E \in \mathscr{E}$, then $x_s \notin E$;

(iii) if $s = (t, n)$ for some $t \in S$, then $d(x_t, x_s) < 1/2^{\|s\|}$.

Choose $x_\varnothing \in F$ arbitrarily. Suppose x_s and E_s are known for all $s \in S$ with $l(s) \leqslant m$ for some $m \geqslant 0$. Given $s = (t, n)$ of length $m + 1$, let B be the basic neighborhood about x_t of d-radius $1/2^{\|s\|}$, and let

$$\mathscr{E}_s = \{E \in \mathscr{E}\colon x_r \in E \text{ for some } r \text{ with } l(r) < l(s)\}.$$

Since $\mathscr{E}_s$ is countable and $x_t \in F$, $B \cap F - \bigcup \mathscr{E}_s$ is not empty, so there is some $E_s \in \mathscr{E} - \mathscr{E}_s$ and some point $x_s \in B \cap F \cap E_s$. Properties (i)–(iii) are clearly satisfied by x_s and E_s.

We let $Q = \{x_s: s \in S\}$. Property (iii) above ensures that Q is d-totally bounded, and so $K = \mathrm{cl}_X Q$ is compact. Now, for each $E \in \mathscr{E}$, the interior of $E \cap K$ relative to K must be empty; otherwise, by the denseness of Q and since $x_{(s,n)}$ converges to x_s, we would have both x_s and $x_{(s,n)}$ belonging to E, for some s and n, in contradiction with (ii) above. Since $E \cap K$ is also an F_σ-set in K, this implies that each $E \cap K$ is of first category in K. Since K is a Baire space, it follows that K cannot be covered by countably many sets from $\mathscr{E}$. This completes the proof in the case when X is completely metrizable.

If X is analytic, then we can find a complete metric space Z and a continuous surjection $f\colon Z \to X$ with the property that the image of any σ-discrete family in Z has a σ-discrete base (and hence refinement) relative to X. Now $\{f^{-1}(E)\colon E \in \mathscr{E}\}$ is a point-countable cover of Z by F_σ-sets, and so must either have a σ-discrete refinement, or else Z contains a compact set C not covered by countably many sets of the form $f^{-1}(E)$, $E \in \mathscr{E}$. But then, by the properties of f, $\mathscr{E}$ either has a σ-discrete refinement, or the compact set $f(C)$ exists and is not covered by any countable subfamily of $\mathscr{E}$. □

We remark that the above proof makes use of several techniques suggested by **[14]**, some of which G. Koumoullis attributes to D. H. Fremlin.[5]

3. Weakly discrete and extended Borel-additive families. Let $\mathscr{E}$ be a family of subsets of X. Following R. Pol **[17]** we say that $A \subset X$ is *$\mathscr{E}$-discrete* provided, for each $a \in A$, there is an $E_a \in \mathscr{E}$ satisfying $E_a \cap A = \{a\}$; if X is a topological space, we say that $\mathscr{E}$ is *weakly discrete* if every $\mathscr{E}$-discrete set is a σ-discrete set in X. Our interest in weakly discrete families stems from the following.

3.1 Lemma. *If $\mathscr{E}$ is a point-countable weakly discrete family of subsets of a space X and L is a Lindelöf subspace of X, then $L \cap (\bigcup\mathscr{E})$ is covered by a countable subfamily of $\mathscr{E}$.*

Proof. Suppose $L \cap (\bigcup\mathscr{E})$ is not covered by any countable subfamily of $\mathscr{E}$. Then, by induction over the countable ordinals, we can easily define a set $A = \{x_\alpha\colon \alpha < \omega_1\}$ contained in L such that, for all $\beta < \alpha$, if $x_\beta \in E \in \mathscr{E}$, then $x_\alpha \notin E$. Then for any $E_\alpha \in \mathscr{E}$ with $x_\alpha \in E_\alpha$, we have $A \cap E_\alpha = \{x_\alpha\}$. Thus A is $\mathscr{E}$-discrete, and so A can be written as a countable union of closed discrete subsets of X. But this implies that L has an uncountable closed discrete subset, contradicting the fact that L is Lindelöf. □

By the *extended Borel sets* of a topological space X we mean the smallest σ-algebra of subsets of X which contains the open sets and is closed to the operation of discrete union **[9]**. By a Souslin set of X we mean, as usual, a subset of X obtained by applying the Souslin operation to the closed sets of X. For an analytic space X it can be shown that the extended Borel sets coincide with the family of all subsets A of X such that A and $X - A$ are Souslin sets of X (see, e.g., **[10]**).

[5] I would like to thank Professor G. Koumoullis for providing a preprint of **[14]**.

We now prove a slight refinement of a result due to R. Pol in the metrizable case [**17**], although the proof given here is considerably less technical.

THEOREM 3.2. *Let X be an analytic space, and let $\mathscr{E}$ be an extended Borel-additive family of subsets of X. Then $\mathscr{E}$ is weakly discrete.*

PROOF. First assume that X is a complete metric space. Let $A \subset X$ be such that, for each $a \in A$, $A \cap E_a = \{a\}$ for some E_a in $\mathscr{E}$. For any nonempty subset B of A, we have

$$B = A \cap \left(\bigcup_{b \in B} E_b \right),$$

so B is a Souslin set relative to A. Thus, if we can show that A is a Souslin set in X, then A will be an analytic metric space all of whose subsets are analytic, and hence a σ-discrete set by the theorem of El′kin. Since

$$A = \bigcup_{a \in A} E_a - \bigcup_{a \in A} E_a \cap (X - \{a\}),$$

we need only show that the set $C = \bigcup_{a \in A} E_a \cap (X - \{a\})$ is extended Borel in X. Since X is metrizable, let $\bigcup_n \mathscr{B}_n$ $(n \in \mathbf{N})$ be an open base for X with each $\mathscr{B}_n$ a discrete family in X. For each B in $\bigcup_n \mathscr{B}_n$ define

$$E_B = \bigcup \{ E_a \colon a \in A \text{ and } B \subset X - \{a\} \},$$

and note that

$$C = \bigcup_n \bigcup_{B \in \mathscr{B}_n} E_B \cap B.$$

Since $\{E_B \cap B \colon B \in \mathscr{B}_n\}$ is a discrete family of extended Borel sets for each n, it follows that C is extended Borel. This proves that $\mathscr{E}$ is weakly discrete when X is completely metrizable.

For X an analytic space, let Z be a complete metric space, and let $f\colon Z \to X$ be a continuous, base-σ-discrete surjection. It is clear that

$$f^{-1}(\mathscr{E}) = \{ f^{-1}(E) \colon E \in \mathscr{E} \}$$

is an extended Borel-additive family in Z. Now let $A \subset X$ and E_a be as before, and let $Z_A = \{z_a \colon a \in A\}$ be such that $z_a \in f^{-1}(a)$ for each a in A. Then Z_A is $f^{-1}(\mathscr{E})$-discrete, and thus a σ-discrete set by the above. But then $A = f(Z_A)$ is σ-discrete, since f is a base-σ-discrete map. □

PROOF OF THEOREM 3.3. Let $\mathscr{E}$ be a point-countable, F_σ-additive family of subsets of the analytic space X, and let $Y = \bigcup \mathscr{E}$. Then Y is an analytic space, and $\mathscr{E}$ is a weakly discrete family in X by Theorem 3.2. In view of Lemma 3.1 and Theorem 2.1, it follows that $\mathscr{E}$ must have a refinement $\mathscr{R}$ that is σ-discrete relative to Y. Since Y is an F_σ-set in X, $\mathscr{R}$ is easily seen to have a σ-discrete refinement relative to X. □

4. Proof of Theorem 4.1. For $n = 1, 2, \ldots$, let $\mathscr{U}_n$ be a locally finite cover of X by open sets having ρ-diameter $< 1/n$. Since each separable subset of X can meet at most countably many members of a locally finite family, our assumptions imply that

$\{F^{-1}(U)\colon U \in \mathscr{U}_1\}$ is a point-countable F_σ-additive cover of T, and so has a σ-discrete refinement $\mathscr{M}$. We may assume that $\mathscr{M}$ is the union of $\mathscr{M}_m$ $(m = 1, 2, \ldots)$, where each $\mathscr{M}_m$ is a discrete family of F_σ-sets in T. Then $M_m = \bigcup \mathscr{M}_m$ is an F_σ-set of T for each m, and, applying the countable reduction principle [**15**, p. 350], there exists a sequence $\{H_m\}_{m \geq 1}$ of pairwise disjoint F_σ-sets such that

$$H_m \subset M_m \quad \text{and} \quad T = \bigcup_{m=1}^{\infty} H_m.$$

(Here we have used the fact that in a regular analytic space, open sets are F_σ-sets, so the above reduction property is valid.) It follows that

$$\mathscr{H} = \{M \cap H_m\colon M \in \mathscr{M}_m, m = 1, 2, \ldots\}$$

is a disjoint, σ-discrete, F_σ-additive refinement of $\{F^{-1}(U)\colon U \in \mathscr{U}_1\}$. For each $H \in \mathscr{H}$ choose some $U_H \in \mathscr{U}_1$ such that $H \subset F^{-1}(U_H)$, and let $H(U) = \bigcup\{H\colon U_H = U\}$ for each $U \in \mathscr{U}_1$. Then the family $\{H(U)\colon U \in \mathscr{U}_1\}$ is also a disjoint, σ-discrete, F_σ-additive cover of T and $H(U) \subset F^{-1}(U)$ for each $U \in \mathscr{U}_1$. Now define the multimap $F_1\colon T \to X$ by

$$F_1(t) = F(t) \cap U \quad \text{iff} \quad t \in H(U),\ U \in \mathscr{U}_1.$$

For any open $V \subset X$, one has

$$F_1^{-1}(V) = \bigcup\{F^{-1}(U \cap V) \cap H(U)\colon U \in \mathscr{U}_1\},$$

and this is an F_σ-set of T as the union of a σ-discrete collection of F_σ-sets. Thus F_1 is weakly F_σ-measurable and has nonempty separable values. It follows that $\{F_1^{-1}(U)\colon U \in \mathscr{U}_2\}$ is a point-countable F_σ-additive cover of T, and we may apply the above argument again to obtain a family $\{K(U)\colon U \in \mathscr{U}_2\}$ that is disjoint, σ-discrete, F_σ-additive, covers T, and is such that $K(U) \subset F_1^{-1}(U)$ for each $U \in U_2$. We proceed to define $F_2\colon T \to X$ by

$$F_2(t) = F_1(t) \cap U \quad \text{iff} \quad t \in K(U),\ U \in \mathscr{U}_2,$$

and observe as before that F_2 is weakly F_σ-measurable and has nonempty separable values. In this way we generate a sequence of weakly F_σ-measurable multimaps $F_n\colon T \to X$ satisfying $F(t) \supset F_1(t) \supset \cdots \supset F_n(t) \supset \cdots$, and $F_n(t)$ is nonempty and has ρ-diameter $< 1/n$.

By the ρ-completeness of the values of F, we can define a map $f\colon T \to X$ by taking $f(t)$ to be the unique member of $\bigcap_n\{\overline{F_n(t)}\}$. Now, for any open $U \subset X$, we have

$$f^{-1}(U) = \bigcup_{n=1}^{\infty} F_n^{-1}(U_n),$$

where $U_n = \{x \in U\colon \rho\text{-dist}(x, X - U) > 1/n\}$. Thus $f^{-1}(U)$ is an F_σ-set of T, proving f is a selector for F of the first class. □

Remark. For recent measurable selection theorems along lines similar to the above, see [**4** and **8**].

We conclude with an example which shows the assumption that F has separable values in Theorem 4.1 cannot be omitted, even when T is completely metrizable.

EXAMPLE. There exists a weakly F_σ-measurable multimap F from the Baire space $B(\omega_1)$ to the space ω_1 with the discrete topology, having no extended Borel measurable selector.

PROOF. Let $B(\omega_1) = \omega_1^{\mathbf{N}}$ with the product topology, and define, for each $\alpha < \omega_1$,

$$S_\alpha = \{x \in B(\omega_1)\colon x(n) \leqslant \alpha \text{ for all } n \in \mathbf{N}\}.$$

It is easy to check that $\{S_\alpha\colon \alpha < \omega_1\}$ is an increasing, F_σ-additive cover of $B(\omega_1)$ by closed, separable subsets. Thus, defining $F\colon B(\omega_1) \to \omega_1$ by

$$F(x) = \{\alpha < \omega_1\colon x \in S_\alpha\},$$

we have $F^{-1}(\alpha) = S_\alpha$, for each α, and so F is weakly F_σ-measurable. Now, if f were an extended Borel measurable selector for F, then $\{f^{-1}(\alpha)\colon \alpha < \omega_1\}$ would be a disjoint extended Borel-additive family in $B(\omega_1)$, and so σ-discrete by [**6**, Theorem 2]. Since this family refines $\{S_\alpha\colon \alpha < \omega_1\}$, it would follow that the latter has a σ-discrete refinement and, hence, that $B(\omega_1)$ can be covered by a σ-discrete collection of separable subsets. But this would imply that $B(\omega_1)$ is σ-locally of weight $< \omega_1$, in contradiction to a theorem of A. H. Stone [**20**, 2.1(7)]. □

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