COMPLETELY SEMISIMPLE SEMIGROUPS
AND EPIMORPHISMS

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ABSTRACT. It is proved that a completely semisimple semigroup \( U \) cannot be
properly epimorphically embedded in another semigroup if \( U \) has no infinite
chain of \( J \)-classes.

Let \( U, S \) be semigroups with \( U \subseteq S \). Following Howie and Isbell [8] we say
\( U \) dominates an element \( d \in S \) if for every semigroup \( T \) and all homomorphisms
\( f: S \to T \) and \( g: S \to T \), \( uf = ug \) for all \( u \in U \) implies \( df = dg \). The set of all
elements of \( S \) dominated by \( U \) is called the dominion of \( U \) in \( S \) and is denoted by
\( \text{Dom}(U, S) \). It is easily verified that \( \text{Dom}(U, S) \) is a subsemigroup of \( S \) containing
\( U \). We say \( U \) is closed in \( S \) if \( \text{Dom}(U, S) = U \) and \( U \) is absolutely closed if \( U \) is closed
in every containing semigroup \( S \). A semigroup \( U \) is saturated if \( \text{Dom}(U, S) \neq S \) for
every properly containing semigroup \( S \).

Let \( f: S \to T \) be a morphism of semigroups. Then \( f \) is an epimorphism (epi for
short) if for every pair of morphisms \( g:T \to V \) and \( h:T \to V \), \( fg = fh \) implies
\( g = h \). It is easy to check that a morphism \( f:S \to T \) is epi if and only if the
inclusion \( i:f(S) \to T \) is epi, and if \( U \subseteq S \) then \( \text{Dom}(U, S) = S \) if and only if
\( i:U \to S \) is epi, whereupon we say that \( U \) is epimorphically embedded in \( S \).

Most results concerning semigroup dominions and epimorphisms are based on
the following result.

RESULT 1 (ISBELL’S ZIGZAG THEOREM [7, THEOREM 2.13]). Let \( U \) be a
subsemigroup of a semigroup \( S \) and let \( d \in S \). Then \( d \in \text{Dom}(U, S) \) if and only if
\( d \in U \) or there is a series of factorizations of \( d \) as follows:

\[
d = u_0y_1 = x_1u_1y_1 = x_1u_2y_2 = x_2u_3y_2 = \cdots = x_mu_{2m-1}y_m = x_mu_{2m},
\]

where \( m \geq 1 \), \( u_i \in U \), \( x_i, y_i \in S \) with \( u_0 = x_1u_1, u_{2i-1}y_i = u_{2i}y_{i+1}, x_iu_{2i} =
\]
\( x_{i+1}u_{2i+1} \) \( (1 \leq i \leq m-1) \), \( u_{2m-1}y_m = u_{2m} \).

Such a series of factorizations is called a zigzag in \( S \) over \( U \) with value \( d \), length
\( m \) and spine \( u_0, u_1, \ldots, u_{2m} \).

Semigroup dominions can also be expressed in terms of special semigroup amalgams
[7, Theorem 2.3]. In particular \( U \) is absolutely closed if and only if every
special amalgam with core \( U \) is strongly embeddable in a semigroup.

Most notable results in this area have been to the effect that certain classes
consist entirely of absolutely closed or of saturated semigroups.

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For example, it was shown by Howie and Isbell [8] that right simple semigroups, finite monogenic semigroups and inverse semigroups are absolutely closed. Scheiblich and Moore [10] showed that the total transformation semigroup, $T_x$, is absolutely closed; a result also proved in Shoji [11] and by Hall [1] whose proof also works for the semigroup of partial transformations on a set.

Howie and Isbell [8] proved that any commutative semigroup that satisfies $M_J$, the minimum condition on $J$ classes, is saturated and the same is true of any finite permutative semigroup (a semigroup is permutative if it satisfies a nontrivial permutation identity) [6]. Another result due to the author [3] is that generalized inverse semigroups are saturated (regular semigroups whose idempotents form a normal subband). Finally, a strong result due to Hall and Jones [2] is that every completely semisimple semigroup with a finite number of $J$-classes is saturated; in particular this includes all completely $[0,\infty]$ simple and finite regular semigroups. We prove a stronger result below.

On the other hand the $2 \times 2$ rectangular band is not absolutely closed [8] and the 3-element null semigroup can have an infinite dominion [1]. Furthermore there are semigroups of each of the following types which are not saturated; commutative cancellative semigroups (the injection of the natural numbers into the integers under addition provides an example), subsemigroups of finite inverse semigroups [9], commutative, periodic semigroups [4], and bands [5]. Indeed recently Trotter [12] has constructed a band with a properly epimorphically embedded subband.

The following theorem was announced at the Marquette Semigroup Conference in 1984. The result will appear in the Proceedings of that conference.

**Theorem 2.** A completely semisimple semigroup $U$ is saturated if it has no infinite chain of $J$-classes.

Before we give the proof we make a couple of technical observations concerning zigzags. Let $Z$ be a zigzag as described in Result 1. If the length of $Z$ is minimum amongst all those zigzags in $S$ over $U$ with value $d$, then $x_i, y_i \in S \setminus U$ for all $i = 1, 2, \ldots, m$. Next suppose that $U$ is properly epimorphically embedded in $S$, and that two successive lines of a zigzag $Z$ of minimum length $m$, with value $d \in S \setminus U$, are $x_i u_2 i y_{i+1} = x_{i+1} u_{2i+1} y_{i+1}$ ($1 \leq i \leq m - 1$). Then, since $y_{i+1} \in S \setminus U$, we have, by the Zigzag Theorem, $y_{i+1} = ay_{i+1}'$ for some $a \in U$, $y_{i+1}' \in S \setminus U$. We may then construct a modified zigzag $Z'$, with value $d$, where the two given lines are replaced by $x_i u_2 i y_{i+1} = x_{i+1} u_{2i+1} y_{i+1}'$ because the necessary equalities are provided by $u_{2i-1} y_i = u_{2i} y_{i+1}'$, $x_i u_{2i} a = x_{i+1} u_{2i+1} a$ and $u_{2i+1} y_{i+1}' = u_{2i+2} y_{i+1}'$. A similar remark also applies to the two initial lines of the zigzag: $u_0 y_1 = x_1 u_1 y_1$. We call the process of passing from $Z$ to $Z'$ “expansion of $Z$ at $y_{i+1}$ via the factorization $y_{i+1} = ay_{i+1}'$”. Of course we can perform this modification for every $y_i$ if we choose. There is a dual comment which applies to the $x_i$ which is also true.

It will be understood throughout that Green's relations are relations in $U$.

**Proof of Theorem 2.** Suppose that $U$ is a completely semisimple semigroup with no infinite chain of $J$-classes properly epimorphically embedded in $S$.

Take $d \in S \setminus U$. Let $C_d$ be the collection of all $J$-classes $J$, such that some $u \in J$ is the first spine member of some zigzag of minimum length in $S$ over $U$ with value
d. Denote the collection of all minimal members of $C_d$ by $\mathcal{C}_d$ and put

$$J_R = \bigcup_{d \in S \setminus U} \mathcal{C}_d.$$ 

The dual collection of $J$-classes will be denoted by $J_L$.

Take $J$ to be a maximal member of $J_R \cup J_L$. Without loss we assume that $J \in J_R$, so that there exists $d \in S \setminus U$ such that $d = u_0y_1$, say, is the first line of the zigzag $Z$ of minimum length $m$ in $S$ over $U$ with value $d$, $u_0 \in J$, and if $d = uy$ is the first line of another such zigzag, then $J_u \neq J$. The maximality condition on $J$ guarantees that the $J$-class of the first spine member of $Z$ is invariant under any expansion. We may also assume that the $J$-class corresponding to each particular spine member is invariant under expansion at $x_i$ or $y_i$ ($1 \leq i \leq m$), because we may expand $Z$ at each $x_i$ or $y_i$ until the $J$-class of each spine member is fixed under any further expansions, which must occur as $U$ satisfies the descending chain condition on $J$-classes. We shall assume that this process has been carried out.

We show that $u_0, u_1, \ldots, u_{2m-1} \in J$. We have $u_0 \in J$, so assume inductively that $u_0, u_1, \ldots, u_{2i} \in J$ ($0 \leq i \leq m - 1$). We first prove that $J \leq J_{u_{2i+1}}$. First if $i = 0$ the equality $u_1 = x_1u_1$ implies that $u_0 = u_0u_1u_1$ ($u_1' \in V(u_1)$), whence $J \leq J_{u_1}$. If $i > 0$ we factorize $x_i$ as $x_i'\alpha_i$, where $\alpha_i \in J_L$. Hence $a_i = \alpha_iu_2t_i$ for some $t_i \in U^1$, whence we have

$$x_i = x_i'\alpha_i = x_i'\alpha_iu_2t_i = x_iu_2t_i = x_{i+1}u_{2i+1}t_i.$$ 

Consider the zigzag $Z'$ which results from expansion of $Z$ at $x_i$ via the factorization $x_i = x_{i+1}u_{2i+1}t_i$. Since the $J$-class of each spine member of $Z$ is invariant under any expansion we obtain

$$J = J_{u_{2i}} = J_{u_{2i+1}}u_{2i+1}u_{2i} \leq J_{u_{2i+1}},$$

as asserted.

Next factorize $y_{i+1}$ as $b_{i+1}y_{i+1}'$ with $b_{i+1} \in J_R$. This gives

$$J \leq J_{u_{2i+1}} = J_{u_{2i+1}b_{i+1}} \leq J_{b_{i+1}},$$

and once again strict inequality is impossible because of the maximality condition on $J$. Therefore $J = J_{u_{2i+1}}$.

The dual argument now establishes that given $u_0, u_1, \ldots, u_{2i+1} \in J$ ($0 \leq i \leq m - 2$), then $u_{2i+2} \in J$. Thus we have proved that $u_0, u_1, \ldots, u_{2m-1} \in J$. Arguing as before we obtain $y_m = b_my_m'$ with $b_m \in J$ and $b_m = s_my_{2m-1}b_m$ for some $s_m \in U^1$. But then

$$y_m = b_my_m' = s_my_{2m-1}b_my_m' = s_my_{2m-1}y_m = s_my_{2m} \in U^1$$

which is a contradiction as $y_m \in S \setminus U$. This completes the proof.

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