A NOTE ON MINIMAL MODULAR SYMBOLS

AVNER ASH

ABSTRACT. For any arithmetic group, a set of geometrically-defined cohomology classes is constructed which spans the cohomology of the group with rational coefficients in the highest nonvanishing dimension thereof.

This note gives a geometrical spanning set $S$ for the cohomology of an arithmetic group in the highest nonvanishing dimension. To do this, I generalize the first three sections of [1]. Unfortunately, I know of no way to generalize the algorithm in §4 of [1]. This means that while the $S$ I construct is infinite, and although a priori the cohomology is finite dimensional, I cannot identify a finite generating set inside $S$, except for the cases covered in [1].

Let $G$ be a semisimple algebraic group defined over $\mathbb{Q}$, $K$ a maximal compact subgroup of $G(\mathbb{R})$, $X = G(\mathbb{R})/K$, and $\Gamma$ an arithmetic subgroup of $G(\mathbb{Q})$. The action of $G(\mathbb{R})$, and hence of $\Gamma$, on $X$ is on the left. It is well known that $H^*(\Gamma, \mathbb{Q}) \simeq H^*(X/\Gamma, \mathbb{Q})$, and by [2] that $H^*(\Gamma, \mathbb{Q}) = 0$ for $* > N$, where

$$N = \dim X - \text{rank}_{\mathbb{Q}}(G).$$

To construct $S \subset H^N(X/\Gamma, \mathbb{Q})$ we proceed as follows: Let $T$ be a maximal $\mathbb{Q}$-split torus of $G$. Without loss of generality, we may assume that $T$ is stable under the Cartan involution of $G$ corresponding to $K$. Set $A = T(\mathbb{R})^0$. Thus $A \cong (\mathbb{R}^+_x)^l$, where $l = \text{rank}_{\mathbb{Q}}(G)$.

Now let $e \in X$ be the base-point corresponding to $K$. Let $\overline{X}$ be the Borel-Serre bordification of $X$ [2]. From the construction of $\overline{X}$ given in [2], it is easy to verify the following:

**Lemma 1.** The closure $Z$ of $Ae$ in $\overline{X}$ is homeomorphic to a ball of dimension $l$. The boundary of $Z$ lies in $\partial \overline{X}$.

Now fix an orientation on $Z$.

**Lemma 2.** The fundamental class of $\partial Z$ freely generates $H_{l-1}(\partial \overline{X}, \mathbb{Z})$ as a $\mathbb{Z}[U(\mathbb{Q})]$-module, where $U$ is the unipotent radical of a minimal $\mathbb{Q}$-parabolic subgroup $P$ of $G$ containing $T$.

The proof of this lemma is essentially contained in §8 of [2]. The central object is the Tits building $B$ of the $\mathbb{Q}$-group $G$. The homotopy equivalence $g_2: B \to \partial \overline{X}$ constructed in [2, 8.4.3] maps one of the apartments of $B$ homeomorphically onto $\partial Z$, if we choose $x$ to be the identity coset $e$ in $X$. Call this distinguished apartment $A_0$. Then choosing an $(l-1)$-dimensional simplex $s$ of $A_0$ corresponds to the choice of $P$ in the lemma. Each apartment is homeomorphic to an $(l-1)$-sphere and $B$...
is homotopy equivalent [2, 8.5.2] to the wedge of all the apartments of $B$ that contain $s$. Hence $\partial X$ is homotopy-equivalent to a wedge of spheres, and [2, 8.6.8] implies that $H_{l-1}(\partial X, \mathbb{Z})$ is a free $\mathbb{Z}[U(\mathbb{Q})]$-module of rank 1, generated by the fundamental class of $g_e(A_0) = \partial Z$.

We shall use the notation $[I]$ for the fundamental class of $\partial Z$ in $H_{l-1}(\partial X, \mathbb{Z})$, where $I$ stands for the identity in $G(\mathbb{Q})$. More generally, if $M$ is any element of $G(\mathbb{Q})$, set $[M]$ to be $M \cdot [I]$, i.e. the fundamental class of $\partial(MZ)$. We call $[M]$ a universal minimal modular symbol. (**Minimal** refers to the fact that it is associated with a minimal $\mathbb{Q}$-parabolic subgroup of $G$.)

Now let $\pi$ denote the canonical projection of $X$ onto $\Gamma\backslash X$. From now on, assume $l \geq 1$. Since $X$ is contractible, $H_{l-1}(\partial X, \mathbb{Z})$ is isomorphic to $H_l(\mathbb{X}, \partial\mathbb{X}, \mathbb{Z})$. Now define $[M]_\Gamma$, for any $M$ in $G(\mathbb{Q})$, to be the image of $[M]$ in $H_l(\Gamma\backslash X, \partial(\Gamma\backslash X), \mathbb{Z})$ under the composition

$$H_{l-1}(\partial X, \mathbb{Z}) \xrightarrow{\sim} H_l(\mathbb{X}, \partial\mathbb{X}, \mathbb{Z}) \xrightarrow{\pi_*} H_l(\Gamma\backslash X, \partial, \mathbb{Z}).$$

It follows from Lemma 2.7 of [4] that $\pi(M(\mathbb{A}_e))$ is a submanifold (with boundary) of $\Gamma\backslash X$. Then $[M]_\Gamma$ is its fundamental class.

**Theorem.** If $\Gamma$ is torsion-free, the modular symbols $[M]_\Gamma$ as $M$ runs through $U(\mathbb{Q})$, or a fortiori through $G(\mathbb{Q})$, generate $H_l(\Gamma\backslash X, \partial, \mathbb{Z})$.

**Proof.** This follows from the fact that $\pi_*$ is surjective. Although the proof of this fact is done in a special case in [1], the argument carries over verbatim.

**Remark.** By Poincaré duality, $H_l(\Gamma\backslash X, \partial, \mathbb{Z})$ is naturally isomorphic to $H^N(\Gamma\backslash X, \mathbb{Z})$ when $\Gamma$ is torsion-free, so we may take the promised set $S$ to be the set of currents corresponding to the modular symbols $[M]_\Gamma, M$ in $U(\mathbb{Q})$.

If $\Gamma$ is not torsion-free, the theorem and remark remain true if $\mathbb{Z}$-coefficients are replaced by $\mathbb{Q}$-coefficients (cf. remark, p. 246 of [1]).

**Example.** If $F$ is any finite extension of $\mathbb{Q}$, let $G$ be the $\mathbb{Q}$-group such that $G(\mathbb{Q}) = SL(n, F), n \geq 2$. Then

$$G(\mathbb{R}) = SL(N, F \otimes \mathbb{Q} \mathbb{R}).$$

View $\mathbb{R}$ as embedded in the natural way in $F \otimes \mathbb{R}$. Then $A$ may be taken to be the diagonal matrices in $G(\mathbb{R})$ with entries in $\mathbb{R}_+^N$, and $l = n - 1$. In case the ring of integers in $F$ is Euclidean, the algorithm in §4 of [1] shows how to reduce $S$ to a finite spanning set. But otherwise, a general algorithm for this problem is unknown and would be of great interest.

**Remarks.** (1) Nonminimal modular symbols may be defined in other dimensions, as in [4]. For these symbols, no such assertion that they generate the cohomology in their dimension is known to be true, nor is any counterexample known to me.

(2) The set $S = \{[M]_\Gamma : M \in U(\mathbb{Q})\}$ depends upon the choices of $T$, $K$ and $U$. We may enlarge $S$ to obtain a set which does not depend on these choices, namely $\{[M]_\Gamma : M \in G(\mathbb{Q})\}$. 
REFERENCES


DEPARTMENT OF MATHEMATICS, OHIO STATE UNIVERSITY, COLUMBUS, OHIO 43210