REMARKS ON PETTIS INTEGRABILITY

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ABSTRACT. Characterizations of Pettis integrability, including the Geitz-Talagrand core theorem, are derived in an easy way.

The purpose of this note is to point out how a folklore result (Proposition 1) can be made the basis for relatively easy proofs of some recent results about Pettis integrability. Our notation follows Dunford and Schwartz [1].

Let $(\Omega, \Sigma, \lambda)$ be a complete probability space, and let $X$ be a Banach space with continuous dual $X^*$. A function $f: \Omega \to X$ is Dunford integrable provided the composition $T(x^*) = x^* f$ is in $L^1(\lambda)$ for every $x^*$ in $X^*$. In this case, it follows (from the closed graph theorem) that $T: X^* \to L^1(\lambda)$ is a bounded linear operator. Hence, for every $g$ in $L^\infty(\lambda)$, the map $\varphi_g$, defined by

$$\varphi_g(x^*) = \int g T(x^*) \, d\lambda,$$

is in $X^{**}$. In particular, for each $E$ in $\Sigma$, $\nu(E) = \int_E f \, d\lambda$, defined to equal $\varphi_{XE}$ and called the Dunford integral of $f$ over $E$, is an element of $X^{**}$.

The function $\nu: \Sigma \to X^{**}$ is not necessarily countably additive. It can be shown that $\nu$ is countably additive if and only if $T$ is a weakly compact operator if and only if $\{x^* f: \|x^*\| \leq 1\}$ is uniformly integrable in $L^1(\lambda)$ [1, pp. 319, 485, 292]. These conditions are automatically satisfied if $f$ has bounded range.

Let $\hat{X}$ denote the natural image of $X$ in $X^{**}$. The function $f$ is said to be Pettis integrable if and only if for every $E$ in $\Sigma$, $\nu(E)$ is in $\hat{X}$ (equivalently, $\nu(E)$ is weak* continuous on $X^*$). The following proposition is essentially a reformulation of the definition.

PROPOSITION 1. A Dunford integrable function $f$ is Pettis integrable if and only if the operator $T: X^* \to L^1(\lambda)$ is weak*-to-weak continuous.

In particular, if $f$ is Pettis integrable then $T$ is necessarily a weakly compact operator.

PROOF. $(\Leftarrow)$ is clear.

$(\Rightarrow)$ If $f$ is Pettis integrable, then for each simple function $g$ in $L^\infty(\lambda)$, $\varphi_g$ is weak* continuous on $X^*$. By approximation, $\varphi_g$ is weak* continuous for every $g$ in $L^\infty(\lambda)$. \hfill \Box

Therefore, to study Pettis integrability one studies the action of $T$ on weak* neighborhoods in $X^*$. If $F$ is a finite set in $X$, and $\varepsilon > 0$, let

$$K(F, \varepsilon) = \{x^* \in X^*: \|x^*\| \leq 1 \text{ and } x^*(x) \leq \varepsilon \text{ for every } x \in F\}.$$
LEMMA 2. If f is Dunford integrable, then for all F, ε the set \( T(\mathcal{K}(F, \varepsilon)) \) is closed and convex in \( L^1(\lambda) \).

PROOF. Convexity is clear. Suppose g is in the closure of \( T(\mathcal{K}(F, \varepsilon)) \), and choose \( x^n_\varepsilon \) in \( \mathcal{K}(F, \varepsilon) \) with \( x^n_\varepsilon f \to g \) a.e. Let \( x^* \) be a weak* cluster point of \( (x^n_\varepsilon)_n \). Then \( x^* \) is in \( \mathcal{K}(F, \varepsilon) \) and \( g = x^* f \) a.e. □

The following reformulation of Proposition 1 was derived from ideas in proofs due to M. Talagrand (see Sentilles and Wheeler [5]).

PROPOSITION 3. If f is Dunford integrable, then the following are equivalent:
1. f is Pettis integrable;
2. T is a weakly compact operator and

\[ \{0\} = \bigcap \{T(\mathcal{K}(F, \varepsilon))|F \subset X, F \text{ finite, and } \varepsilon > 0\}. \]

PROOF. (1) \( \Rightarrow \) (2) If f is Pettis integrable, then T is weakly compact. Suppose g is in \( \{T(\mathcal{K}(F, \varepsilon))|F \subset X, F \text{ finite} \} \). For each \( (F, \varepsilon) \) choose \( x^*_{F, \varepsilon} \) in \( \mathcal{K}(F, \varepsilon) \) so that \( g = T(x^*_{F, \varepsilon}) \). Note that \( (x^*_{F, \varepsilon})_{(F, \varepsilon)} \) is naturally a net in \( X^* \) which converges weak* to 0. Hence, \( g = T(x^*_{F, \varepsilon}) \to 0 \).

(2) \( \Rightarrow \) (1) Let \( B^* = \{x^*||x^*| \leq 1\} \). Suppose a net \( (x^*_\alpha) \) in \( (1/2)B^* \) converges weak* to \( x^* \). Then \( (x^*_\alpha - x^*) \) is in \( B^* \) and for all \( (F, \varepsilon) \) it is eventually in \( \mathcal{K}(F, \varepsilon) \). Let \( g \) be any weak cluster point of \( (T(x^*_\alpha - x^*)) \). Then \( g \) is in \( \bigcap_{(F, \varepsilon)} T(\mathcal{K}(F, \varepsilon)) \), so \( g = 0 \). Thus \( T(x^*_\alpha) \to T(x^*) \) weakly in \( L^1(\lambda) \). It follows that T is weak*-to-weak continuous. □

Say that a weakly measurable function \( f: \Omega \to X \) is separable-like provided there exists a separable subspace D of X such that for every \( x^* \) in \( X^* \),

\[ x^* \chi_D f = x^* f \] a.e. (\( \lambda \)).

(That is, as far as \( x^* \) is concerned, \( f \) takes almost all its range in \( D \).) In particular, simple functions are separable-like. If \( (\Omega, \Sigma, \mu) \) is a separable measure space, then every Dunford integrable function is automatically separable-like.

COROLLARY 4. Suppose f is Dunford integrable and T is weakly compact. If f is separable-like, then it is Pettis integrable.

PROOF. Let \( (x_n) \) be dense in D. Let g be in \( \bigcap_{(F, \varepsilon)} T(\mathcal{K}(F, \varepsilon)) \). We must show that \( g = 0 \) a.e.

For each \( n \), choose \( x^n_\varepsilon \) in \( \mathcal{K}(\{x_i\}_{i=1}, 1/n) \) so that \( g = x^n_\varepsilon f \) a.e. Now choose a fixed null set E so that for every \( n \), \( g = x^n_\varepsilon f \) off E. Let \( (x^*_n)_n \) cluster weak* at \( x^* \). Then \( g = x^* f \) off E, while \( x^* = 0 \) on D. Hence,

\[ g = x^* f = x^* \chi_D f = 0 \] a.e. □

If \( (\Omega, \Sigma, \mu) \) is a perfect measure space, Geitz [3] shows that every Pettis integrable f is separable-like. Thus, the converse of the Corollary holds for perfect measure spaces.

The next corollary is obvious.

COROLLARY 5. Suppose f is Dunford integrable, T is weakly compact, and there is a sequence \( (f_n) \) of separable-like integrable functions such that for each \( x^* \), \( (x^* f_n) \) converges a.e. to \( x^* f \). Then f is Pettis integrable.
If \( f: \Omega \to X \) is Dunford integrable and \( T \) is weakly compact, then \( f \) is Pettis integrable if and only if

\[
\int_E f \, d\lambda = \varnothing \quad \text{if} \quad \lambda(E) = 0.
\]

PROOF. (\( \Rightarrow \)) If \( f \) is Pettis integrable, then by the separation theorem the integral \( \int_E f \, d\lambda \) is in \( \varnothing \).

(\( \Leftarrow \)) Suppose \( (*) \) holds and \( g \) is in \( \bigcap_{(F,\varepsilon)} T(\mathcal{K}(F,\varepsilon)) \), with \( g = x^* f \) for some \( x^* \) in \( X^* \). If \( g \) is not identically zero a.e., then there exists \( x \) in \( \varnothing \) with \( x^*(x) \neq 0 \).

For each \( n \), choose \( x^*_n \) in \( \mathcal{K}(\{x\}, 1/n) \) with \( g = x^*_n f \) a.e. Choose a fixed null set \( E \) so that for every \( n \), \( g = x^*_n f \) off \( E \). Let \( y^* \) be a weak* cluster point of \( (x^*_n) \). Then \( y^* f = g \) a.e., and \( y^*(x) = 0 \).

Let \( z^* = x^* - y^* \). Then \( z^* f = 0 \) a.e. while \( z^*(x) \neq 0 \), contradicting the lemma.

REFERENCES


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