SOME SPECTRAL PROPERTIES OF THE PERTURBED POLYHARMONIC OPERATOR

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ABSTRACT. We deal with the polyharmonic operator perturbed by a potential, decreasing at infinity as $|x|^{-\sigma}$. Under some conditions we obtain the absence of eigenvalues in a neighbourhood of the point $z = 0$, the existence of the strong limit and the asymptotic expansion of the corresponding resolvent $R_z$, considered in weighted $L^2$-spaces, as $z \to 0$, where $z$ is the spectral parameter.

Let us consider the operator

$$L = (-\Delta)^m + q(x)$$

in the space $L^2(\mathbb{R}^n)$, where $n$ is odd,

$$2m > n,$$

$\text{Im } q = 0$, $q$ is Lebesgue measurable on $\mathbb{R}^n$, a.e. on $\mathbb{R}^n$,

$$|q| \leq c(1 + |x|)^{-\sigma},$$

where $c$ does not depend on $x$. The operator $L$ defined on the Sobolev space $H_{2m}(\mathbb{R}^n)$ is selfadjoint. Let $Q$ be the set of functions $h$, defined on $\mathbb{R}^n$, each of them vanishing a.e. outside a sphere depending on $h$. It is known [3] that if

$$q \in Q$$

then the resolvent kernel $G(x, y, k)$ of $L$, where

$$k = z^{1/2m} \in S_m := \{k: k \in \mathbb{C}, 0 < \arg k < \pi/m\},$$

has a meromorphic continuation onto the whole $\mathbb{C}$. In the special case, where $q = 0$ on $\mathbb{R}^n$ and (2) is valid, the point $k = 0$ is a pole of $G$. But if a.e. on $\mathbb{R}^n$,

$$q \geq 0,$$

and (5) is valid, then $[1] k = 0$ is a regular point of $G$. We shall investigate the behaviour of $R_z$ as $z \to 0$ in the case where (5) does not hold, and therefore $G$ in general has no analytical continuation outside $S_m$. The results obtained below
may be applied to the asymptotics as \( t \to \infty \) of solutions of the corresponding nonstationary problems (see e.g. [2]).

Let \( \phi \) be the class of functions \( v \in H_{2m}^{\text{loc}} \) satisfying the following conditions as \( \rho \to \infty \):

\[
\int_{|x| = \rho} |D^j v|^2 \, dx = O(\rho^{2\kappa}),
\]

where \( D^j \) is any derivative of order \( |j|, 0 \leq |j| \leq 2m - 1 \), and

\[
\kappa = m - |j| - 1.
\]

**Condition (U).** The equation

\[
(-\Delta)^m v + q(x)v = 0
\]

has only the trivial solution in the class \( \phi \).

It is known that (2), (6), (7) imply (U). Denote by \( N_\varepsilon \) the circle \( |k| < \varepsilon \) in \( \mathbb{C} \), and set

\[
N(\varepsilon) = N_\varepsilon \cap S_m, \quad N^+(\varepsilon) \cup (0, \varepsilon),
\]

where \((0, \varepsilon)\) is the interval \( 0 < k < \varepsilon \). Let \( L^2_s \) be the space of functions, \( \varphi \), defined on \( \mathbb{R}^n \), with the norm

\[
\| \varphi \|^2_{L^2_s} = \int (1 + |x|)^s |\varphi|^2 \, dx
\]

(we integrate over \( \mathbb{R}^n \)), and let \( B_s \) be the normed space of bounded operators \( A : L^2_s \to L^2_{-s} \), where \( s > 0 \).

**THEOREM 1.** Let conditions (U), (2), (3), (4) be satisfied. Then there exists \( \varepsilon > 0 \) such that

(i) There are no eigenvalues of \( L \) in the interval \((-\varepsilon^{2m}, \varepsilon^{2m})\).

(ii) For every

\[
s = 4m - n + \delta,
\]

where \( \delta > 0 \), it is possible to find a constant \( c_s > 0 \) so that the inequality

\[
\| R_z \|_{B_s} \leq c_s
\]

holds for each \( k \in N(\varepsilon) \), where \( z = k^{2m} \).

**PROOF.** Choose some function \( q_0(x) \) continuous on \( \mathbb{R}^n \) and satisfying conditions (5), (6), (7), and set \( L_0 = (-\Delta)^m + q_0 \). It is known [1] that the corresponding resolvent kernel \( G_0(x, y, k) \) is continuous in some domain \( \mathbb{R}^n \times \mathbb{R}^n \times N_{\varepsilon'}, \varepsilon' > 0 \), holomorphic on \( N_{\varepsilon'} \) in \( k \) and for \( x, y \in \mathbb{R}^n, k \in N(\varepsilon') \),

\[
|G_0(x, y, k)| \leq c(XY)^{m-n},
\]

where \( X = 1 + |x|, Y = 1 + |y|, \) and \( c \) does not depend on \( x, y, k \). Choose some \( \delta > 0 \) such that

\[
\delta < 2(\sigma - (4m - n))
\]

Suppose (i) is not valid. Then there exists a sequence \( z_l = k_l^{2m} \to 0, k_l \in N^+(\varepsilon') \), and a sequence \( v_l \in H_{2m}(\mathbb{R}^n) \) so that

\[
\| v_l \|_{-s} = 1,
\]

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where $s$ is defined by (11), (14), and a.e. on $\mathbb{R}^n$,

$$v_l = \int G_0(x, y, k)q_1(y)v_l(y)\,dy,$$

where $q_1 = q_0 - q$. Since (3), (11), (14), (15), (16) a.e. on $\mathbb{R}^n$,

$$|v_l| \leq cX^{2m-n},$$

$l = 1, 2, \ldots$. Because of (13), (15), (16), (17), there exists a subsequence $v_{l_j}$ converging in $L^2_{-s}$ to $v$, and a.e. on $\mathbb{R}^n$,

$$v(x) = \int G_0(x, y, 0)q_1(y)v(y)\,dy.$$

It follows from (18) (see [1]) that $v \in \phi$. Because of (10), (U) we conclude that $v = 0$ a.e. on $\mathbb{R}^n$. This contradicts (15); therefore (i) is proved. In order to prove (ii), suppose there exist some $s$, defined by (11) and (14), $f \in L^2_s$ and a sequence $k_l \to 0$, $k_l \in S_m$, so that $\|u_l\|_{-s} \to \infty$, where $u_l = R_{z_l}f$, $z_l = k_l^{2m}$. Set $v_l = \|u_l\|^{-1}_{-s}u_l$. As above we obtain $v_{l_j} \to 0$ in $L^2_{-s}$, which contradicts (15). So the proof is complete.

**COROLLARY.** *Let the conditions of Theorem 1 be satisfied. Then the strong limit $\rho$ of $R_z: L^2_s \to L^2_s$ does exist, where $s$ is defined by (11), as $z \to 0$, $\operatorname{Im} z \neq 0$. Moreover, $\rho f \in \phi$ for any $f \in L^2_s$.***

The next theorem immediately follows from Theorem 1 and the results of M. Murata [2, Theorems 8.7, 8.9, 8.10].

**THEOREM 2.** *Let conditions (U), (2), (3) be satisfied, where $\sigma > 4m - n + 2l + 2$, $l \geq 0$ is an integer. Then for each number $s > 4m - n + 2l + 2$ there exist operators $\rho_j \in B_s$, $j = 0, 1, \ldots, l$ such that*

$$R_z = \sum_{j=0}^{l} \rho_j k^j + g(k)k^l,$$

*where $z = k^{2m}$, $k \in S_m$, and $\|g(k)\|_{B_s} \to 0$ as $k \to 0$. Moreover if $j/2m$ is not an integer, then $\rho_j$ is a finite rank operator.*

**REFERENCES**


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