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SOME SPECTRAL PROPERTIES OF THE PERTURBED POLYHARMONIC OPERATOR

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ABSTRACT. We deal with the polyharmonic operator perturbed by a potential, decreasing at infinity as $|x|^{-\sigma}$. Under some conditions we obtain the absence of eigenvalues in a neighbourhood of the point $z = 0$, the existence of the strong limit and the asymptotic expansion of the corresponding resolvent R_z, considered in weighted L^2-spaces, as $z \to 0$, where z is the spectral parameter.

Let us consider the operator

$$L = (-\Delta)^m + q(x) \tag{1}$$

in the space $L^2(\mathbf{R}^n)$, where n is odd,

$$2m > n, \tag{2}$$

$\operatorname{Im} q = 0$, q is Lebesgue measurable on $\mathbf{R}^n$, a.e. on $\mathbf{R}^n$,

$$|q| \leq c(1 + |x|)^{-\sigma}, \tag{3}$$

$$\sigma > 4m - n, \tag{4}$$

and c does not depend on x. The operator L defined on the Sobolev space $H_{2m}(\mathbf{R}^n)$ is selfadjoint. Let Q be the set of functions h, defined on $\mathbf{R}^n$, each of them vanishing a.e. outside a sphere depending on h. It is known [**3**] that if

$$q \in Q \tag{5}$$

then the resolvent kernel $G(x, y, k)$ of L, where

$$k = z^{1/2m} \in S_m := \{k : k \in \mathbf{C}, 0 < \arg k < \pi/m\},$$

has a meromorphic continuation onto the whole $\mathbf{C}$. In the special case, where $q = 0$ on $\mathbf{R}^n$ and (2) is valid, the point $k = 0$ is a pole of G. But if a.e. on $\mathbf{R}^n$,

$$q \geq 0, \tag{6}$$

$$\operatorname{meas}\{x : x \in \mathbf{R}^n, q(x) > 0\} > 0 \tag{7}$$

and (5) is valid, then [**1**] $k = 0$ is a regular point of G. We shall investigate the behaviour of R_z as $z \to 0$ in the case where (5) does not hold, and therefore G in general has no analytical continuation outside S_m. The results obtained below

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may be applied to the asymptotics as $t \to \infty$ of solutions of the corresponding nonstationary problems (see e.g. [**2**]).

Let ϕ be the class of functions $v \in H_{2m}^{\mathrm{loc}}$ satisfying the following conditions as $\rho \to \infty$:

$$\int_{|x|=\rho} |D^j v|^2\,dx = O(\rho^{2\kappa}), \tag{8}$$

where D^j is any derivative of order $|j|$, $0 \le |j| \le 2m-1$, and

$$\kappa = m - |j| - 1. \tag{9}$$

Condition (U). The equation

$$(-\Delta)^m v + q(x)v = 0 \tag{10}$$

has only the trivial solution in the class ϕ.

It is known that (2), (6), (7) imply (U). Denote by N_ε the circle $|k| < \varepsilon$ in $\mathbf{C}$ and set

$$N(\varepsilon) = N_\varepsilon \cap S_m, \qquad N^+(\varepsilon) \cup (0,\varepsilon),$$

where $(0,\varepsilon)$ is the interval $0 < k < \varepsilon$. Let L_s^2 be the space of functions, φ, defined on $\mathbf{R}^n$, with the norm

$$\|\varphi\|_s^2 = \int (1+|x|)^s |\varphi|^2\,dx$$

(we integrate over $\mathbf{R}^n$), and let B_s be the normed space of bounded operators $A\colon L_s^2 \to L_{-s}^2$, where $s > 0$.

THEOREM 1. *Let conditions* (U), (2), (3), (4) *be satisfied. Then there exists* $\varepsilon > 0$ *such that*

(i) *There are no eigenvalues of L in the interval* $(-\varepsilon^{2m}, \varepsilon^{2m})$.

(ii) *For every*

$$s = 4m - n + \delta, \tag{11}$$

where $\delta > 0$, it is possible to find a constant $c_s > 0$ so that the inequality

$$\|R_z\|_{B_s} \le c_s \tag{12}$$

holds for each $k \in N(\varepsilon)$, where $z = k^{2m}$.

PROOF. Choose some function $q_0(x)$ continuous on $\mathbf{R}^n$ and satisfying conditions (5), (6), (7), and set $L_0 = (-\Delta)^m + q_0$. It is known [**1**] that the corresponding resolvent kernel $G_0(x,y,k)$ is continuous in some domain $\mathbf{R}^n \times \mathbf{R}^n \times N_{\varepsilon'}$, $\varepsilon' > 0$, holomorphic on $N_{\varepsilon'}$ in k and for $x, y \in \mathbf{R}^n$, $k \in N(\varepsilon')$,

$$|G_0(x,y,k)| \le c(XY)^{2m-n}, \tag{13}$$

where $X = 1 + |x|$, $Y = 1 + |y|$, and c does not depend on x, y, k. Choose some $\delta > 0$ such that

$$\delta < 2(\sigma - (4m - n)). \tag{14}$$

Suppose (i) is not valid. Then there exists a sequence $z_l = k_l^{2m} \to 0$, $k_l \in N^+(\varepsilon')$, and a sequence $v_l \in H_{2m}(\mathbf{R}^n)$ so that

$$\|v_l\|_{-s} = 1, \tag{15}$$

where s is defined by (11), (14), and a.e. on $\mathbf{R}^n$,

$$v_l = \int G_0(x, y, k) q_1(y) v_l(y)\, dy, \tag{16}$$

where $q_1 = q_0 - q$. Since (3), (11), (14), (15), (16) a.e. on $\mathbf{R}^n$,

$$|v_l| \le cX^{2m-n}, \tag{17}$$

$l = 1, 2, \ldots$. Because of (13), (15), (16), (17), there exists a subsequence v_{l_j} converging in L^2_{-s} to v, and a.e. on $\mathbf{R}^n$,

$$v(x) = \int G_0(x, y, 0) q_1(y) v(y)\, dy. \tag{18}$$

It follows from (18) (see [**1**]) that $v \in \phi$. Because of (10), (U) we conclude that $v = 0$ a.e. on $\mathbf{R}^n$. This contradicts (15); therefore (i) is proved. In order to prove (ii), suppose there exist some s, defined by (11) and (14), $f \in L^2_s$ and a sequence $k_l \to 0$, $k_l \in S_m$, so that $\|u_l\|_{-s} \to \infty$, where $u_l = R_{z_l} f$, $z_l = k_l^{2m}$. Set $v_l = \|u_l\|_{-s}^{-1} u_l$. As above we obtain $v_{l_j} \to 0$ in L^2_{-s}, which contradicts (15). So the proof is complete.

COROLLARY. *Let the conditions of Theorem* 1 *be satisfied. Then the strong limit ρ of $R_z\colon L^2_s \to L^2_{-s}$ does exist, where s is defined by* (11), *as* $z \to 0$, $\operatorname{Im} z \ne 0$. *Moreover, $\rho f \in \phi$ for any $f \in L^2_s$.*

The next theorem immediately follows from Theorem 1 and the results of M. Murata [**2**, Theorems 8.7, 8.9, 8.10].

THEOREM 2. *Let conditions* (U), (2), (3) *be satisfied, where* $\sigma > 4m - n + 2l + 2$, $l \ge 0$ *is an integer. Then for each number* $s > 4m - n + 2l + 2$ *there exist operators* $\rho_j \in B_s$, $j = 0, 1, \ldots, l$ *such that*

$$R_z = \sum_{j=0}^{l} \rho_j k^j + g(k) k^l, \tag{19}$$

where $z = k^{2m}$, $k \in S_m$, *and* $\|g(k)\|_{B_s} \to 0$ *as* $k \to 0$. *Moreover if $j/2m$ is not an integer, then ρ_j is a finite rank operator.*

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