ADDITIVITY OF JORDAN*-MAPS ON AW*-ALGEBRAS
JÔSUKE HAKEDA

ABSTRACT. Let M and N be AW*-algebras and φ be a Jordan*-map from
M to N which satisfies
(1) φ(x o y) = φ(x) o φ(y) for all x and y in M,
(2) φ(x*) = φ(x)* for all x ∈ M, and
(3) φ is bijective, where x o y = (1/2)(xy + yx).

If M has no abelian direct summand and a Jordan*-map φ is uniformly
continuous on every abelian C*-subalgebra of M, then we can conclude that
φ is additive. Moreover, φ is the sum of φ_i (i = 1,2,3,4) such that φ_1 is
a linear *-ring isomorphism, φ_2 is a linear *-ring anti-isomorphism, φ_3 is a
conjugate linear *-ring anti-isomorphism and φ_4 is a conjugate linear *-ring
isomorphism.

1. Preliminaries. The special Jordan product (resp. the special Jordan triple
product) is defined by x o y = (1/2)(xy + yx) (resp. {x, y, z} = (1/2)(xyz + zyx)).

DEFINITION 1.1. Let M and N be AW*-algebras, and let φ be a map from M
to N. If φ satisfies the following conditions (i)−(iii), then φ is called a Jordan*-map.
(i) φ(x o y) = φ(x) o φ(y) for x and y of M,
(ii) φ(x*) = φ(x)* for x ∈ M, and
(iii) φ is bijective.

Throughout this paper, we always assume that M and N are AW*-algebras and
φ is a Jordan*-map from M to N.

LEMMA 1.2. Let e and f be projections. Then
(i) ef = 0 if and only if e o f = 0, and
(ii) e = ef if and only if e = e o f.

PROOF. (i) If ef = 0, then e o f = 0 is obvious. Suppose e o f = 0. Since
ef = -ef, ef = (ef)f = (-ef)f = (-ef)ef = (ef)^2 = e(ef)f = e(-ef)f = -ef.
So ef = 0. (ii) By the assertion (i), we have e(1-f) = 0 if and only if e o (1-f) = 0.
Hence e = ef if and only if e = e o f.

COROLLARY 1.3. φ|M_p is a lattice isomorphism between the lattice M_p of pro-
jections of M and the lattice N_p of projections of N and preserves the orthogonality.

COROLLARY 1.4. (i) φ(0) = 0, φ(1) = 1,
(ii) If {e_i: i = 1,2,...,n} ⊂ M_p is an orthogonal family, then φ(Σ_i α_i e_i) =
Σ_i φ(α_i e_i) for {α_i: i = 1,2,...,n} ⊂ C (where C is the complex numbers), and
(iii) If e ≤ f, then φ(f - e) = φ(f) - φ(e) (in particular φ(1-e) = 1 - φ(e)).
The assertions (i) and (iii) are obvious. If \( \{ e_i : i = 1, 2, \ldots, n \} \subset M_p \) is an orthogonal family, then \( \{ \phi(e_i) : i = 1, 2, \ldots, n \} \subset N_p \) is an orthogonal family. Put \( x = \sum_i \alpha_i e_i \). Then

\[
\phi(x) = \phi(x) \circ \left( \bigvee_j e_j \right) = \phi(x) \circ \phi \left( \bigvee_j e_j \right) = \phi(x) \circ \left( \bigvee_j \phi(e_j) \right)
\]

\[
= \phi(x) \circ \left( \sum_j \phi(e_j) \right) = \sum_j \phi(x \circ e_j) = \sum_j \phi(\alpha_j e_j).
\]

**LEMMA 1.5.** The following assertions (i) and (ii) hold.

(i) \( \phi(-x) = -x \) for every \( x \in M \).

(ii) There exists a unique central projection \( e_0 \) of \( M \) such that \( \phi(i \cdot 1) = i\phi(e_0) - i\phi(1 - e_0) \) \( (i^2 = -1) \).

**PROOF.** (i) Put \( e_1 = \phi^{-1}(1/2)(1 + \phi(-1)) \). Then

\[
\phi(-e_1) = \phi(-1) \circ \phi(e_1) = \phi(-1) \circ ((1/2)(1 + \phi(-1)))
\]

\[
= (1/2)(\phi(-1) + \phi(-1) \circ \phi(-1)) = \phi(e_1).
\]

Since \( \phi \) is bijective, we get \( -e_1 = e_1 \) and so \( e_1 = 0 \). Therefore \( \phi(-1) = -1 \) and \( \phi(-x) = \phi(-1) \circ \phi(x) = -\phi(x) \) follow for all \( x \in M \).

(ii) Next, we shall show that \( \phi(i \cdot 1) \) is a central element of \( N \).

\[
\phi(x) = \phi(-((i \cdot 1) \circ x) \circ (i \cdot 1))
\]

\[
= -(1/2)\phi(i \cdot 1)\phi(x)\phi(i \cdot 1) + (1/2)\phi(x)
\]

for every \( x \in M \). Hence \( \phi(x) = -\phi(i \cdot 1)\phi(x)\phi(i \cdot 1) \) and so \( \phi(i \cdot 1)\phi(x) = -\phi(i \cdot 1)^2\phi(x)\phi(i \cdot 1) = \phi(x)\phi(i \cdot 1) \). Therefore \( \phi(i \cdot 1) \) is a central element of \( N \).

Put \( e_0 = \phi^{-1}(1/2)(1 - i\phi(i \cdot 1)) \). Then \( \phi(e_0) = (1/2)(1 - i\phi(i \cdot 1)) \) is a central projection of \( N \). Since

\[
\phi(e_0 \circ f) = \phi(e_0) \circ \phi(f) = \phi(e_0)\phi(f)
\]

\[
= \phi(e_0) \land \phi(f) = \phi(e_0 \land f)
\]

for all \( f \in M_p \),

\[
0 = e_0 \circ f - e_0 \land f = e_0 \circ (f - e_0 \land f) = e_0(f - e_0 \land f)
\]

by Lemma 1.2 (i.e. \( e_0 f = e_0 \land f \in M_p \)). Hence, \( e_0 \) commutes with \( f \), so \( e_0 \) is a central projection of \( M \) and

\[
\phi(i \cdot 1) = i\phi(e_0) - i(1 - \phi(e_0)) = i\phi(e_0) - i\phi(1 - e_0).
\]

Finally, we shall show that \( e_0 \) is unique. Suppose \( \phi(i \cdot 1) = if - i(-1 - f) \) for some projection \( f \) in \( N \). Then \( if = \phi(i \cdot 1)f = i\phi(e_0)f - i\phi(1 - e_0)f \).

Since \( if - i\phi(e_0)f = -i(1 - \phi(e_0))f \), we get \( (1 - \phi(e_0))f = 0 \) and so \( f \leq \phi(e_0) \). Therefore \( f = \phi(e_0) \) by the symmetry of \( \phi(e_0) \) and \( f \).
LEMMA 1.6. \( \phi(\text{exe}) = \phi(e)\phi(x)\phi(e) \) holds for any projection \( e \) in \( M \) and any \( x \) in \( M \).

PROOF. Since \( \phi(2e - 1) = \phi(e) - \phi(1 - e) = 2\phi(e) - 1 \),
\[
\phi(\text{exe}) = \phi(((2e - 1) \circ x) \circ e) = (\phi(2e - 1) \circ \phi(x)) \circ \phi(e)
= ((2\phi(e) - 1) \circ \phi(x)) \circ \phi(e) = \phi(e)\phi(x)\phi(e).
\]

2. \( AW^*\)-algebra which has an \( n \times n \) (\( n \geq 2 \)) matrix unit. Throughout this paper, we suppose that \( M \) has an \( n \times n \) (\( n \geq 2 \)) matrix unit.

LEMMA 2.1. (i) \( \phi|C \cdot 1 \) is additive, (ii) For every \( x \in M \), \( \phi(\rho x) = \rho \phi(x) \) holds for all rational \( \rho \).

PROOF. Let \( \{v_{ij}\} \) be the matrix unit of \( M \). Put \( e = v_{ii}, \, v = v_{ij} \,(i \neq j) \), \( p = (1/2)(e + v^*)(e + v) \) and \( q = (1/2)(e - v^*)(e - v) \). Since \( p \) and \( q \) are orthogonal projections in \( M \), we have
\[
\phi((\alpha + \beta)e) = \phi(e(2\alpha p + 2\beta q)e) = \phi(e)(2\alpha p + 2\beta q)\phi(e)
= \phi(e)(\phi(2\alpha p) + \phi(2\beta q))\phi(e) = \phi(\alpha e) + \phi(\beta e)
\]
by Lemma 1.6 and Corollary 1.4. So our assertion (i) follows. Let \( n \) be an arbitrary integer and \( m \) be a natural number. Then, for every \( x \in M \),
\[
n\phi(x) = \phi(n \cdot 1) \circ \phi(x) = \phi(nx) = \phi(m((n/m)x))
= \phi(m \cdot 1) \circ \phi((n/m)x) = m\phi((n/m)x)
\]
follows. So we have assertion (ii).

COROLLARY 2.2. Let \( e \) and \( f \) be orthogonal projections of \( M \). Then \( \phi(\{e, x, f\}) = \{\phi(e), \phi(x), \phi(f)\} \) holds.

PROOF. Since \( 2(e \circ x) \circ f = \{e, x, f\} \) and \( \phi(e)\phi(f) = 0 \) (Lemma 1.2), we have
\[
\phi(\{e, x, f\}) = \phi(2(e \circ x) \circ f) = 2(\phi(e) \circ \phi(x)) \circ \phi(f)
= \{\phi(e), \phi(x), \phi(f)\}.
\]

LEMMA 2.3. \( \phi(\lambda \cdot 1) = \lambda \cdot 1 \) holds for all \( \lambda \in \mathbb{R} \) (where \( \mathbb{R} \) is the real numbers).

PROOF. Since \( \phi|C \cdot 1 \) is additive, \( \phi(\rho \cdot 1) = \rho \cdot 1 \) for every rational number \( \rho \). Let \([\lambda]\) be the integral part of \( \lambda \in \mathbb{R} \). Then we have
\[
0 \leq \phi(\lambda \cdot 1) \leq \phi([1/\lambda]^{-1} \cdot 1) \leq [1/\lambda]^{-1} \cdot 1 \quad \text{for all } \lambda \in (0, 1).
\]
Since \( \phi(-1) = -1 \), the map \( \lambda \mapsto \phi(\lambda \cdot 1) \) is continuous at 0. Hence, the map is continuous on \( \mathbb{R} \).

Therefore we get \( \phi(\lambda \cdot 1) = \lambda \cdot 1 \) for all \( \lambda \in \mathbb{R} \).

LEMMA 2.4. There exists a unique central projection \( e_0 \) of \( M \) such that \( \phi(\alpha \cdot 1) = \alpha\phi(e_0) + \bar{\alpha}\phi(1 - e_0) \) for any \( \alpha \in \mathbb{C} \).

PROOF. For every \( \lambda, \mu \in \mathbb{R} \),
\[
\phi((\lambda + i\mu) \cdot 1) = \phi(\lambda \cdot 1 + \phi(i \cdot 1)\phi(\mu \cdot 1)
= \lambda \cdot 1 + (i\phi(e_0) - i\phi(1 - e_0))(\mu \cdot 1)
= (\lambda + i\mu)\phi(e_0) + (\lambda - i\mu)\phi(1 - e_0) \quad (i^2 = -1)
\]
by Lemma 2.1 and Lemma 1.5.
**Lemma 2.5.** Let \( a = \sum_i \alpha_i e_i \) where \( \alpha_i \) (\( i = 1, 2, \ldots, n \)) are in \( \mathbb{C} \) and \( e_i \) (\( i = 1, 2, \ldots, n \)) are orthogonal projections such that \( \sum_i e_i = 1 \). Then
\[
\phi(axa) = \phi(a)\phi(x)\phi(a) \quad \text{for all } x \in M.
\]

**Proof.** Since \( \sum_i \phi(e_i) = 1 \) (Corollary 1.4),
\[
\phi(axa) = \left( \sum_i \phi(e_i) \right) \phi(axa) \left( \sum_i \phi(e_i) \right)
\]
\[
= \sum_i \phi(e_i)\phi(axa)\phi(e_i) + 2 \sum_{i<j} \{\phi(e_i), \phi(axa), \phi(e_j)\}
\]
\[
= \sum_i \phi(e_i axae_i) + 2 \sum_{i<j} \phi(\{e_i, axa, e_j\})
\]
\[
= \sum_i \phi(\alpha_i \cdot 1^2)\phi(e_i)\phi(x)\phi(e_i)
\]
\[
+ 2 \sum_{i<j} \phi(\alpha_i \cdot 1)\phi(\alpha_j \cdot 1)\phi(\{e_i, e_j, e_j\})
\]
\[
= \phi(a)\phi(x)\phi(a)
\]
by Lemma 1.6 and Corollary 2.2.

**3. Structure of Jordan*-maps.** In this section we assume that \( \phi \) satisfies the following condition:

(iv) \( \phi \) is uniformly continuous on every abelian C*-algebra of \( M \).

**Lemma 3.1.** If \( h \in M \) is selfadjoint, then \( \phi|C^*(h, 1) \) is additive where \( C^*(h, 1) \) is the C*-subalgebra which is generated by \( h \) and \( 1 \).

**Proof.** Let \( h = \int_{\sigma(h)} \lambda \, d e_\lambda \) be the spectral decomposition of \( h \), where \( \sigma(h) \) is the spectrum of \( h \).

For any \( x \) and \( y \) in \( C^*(h, 1) \), there exist \( f \) and \( g \) in \( C(\mathbb{R}) \) (\( C(\mathbb{R}) \) is the C*-algebra of the complex-valued continuous functions on \( \mathbb{R} \)) such that
\[
x = \int_{\sigma(h)} f(\lambda) \, d e_\lambda = \lim_j \sum_j f(\lambda_j) e_j
\]
and
\[
y = \int_{\sigma(h)} g(\lambda) \, d e_\lambda = \lim_j \sum_j g(\lambda_j) e_j.
\]
So we have
\[
\phi(x + y) = \lim_j \sum_j \phi((f(\lambda_j) + g(\lambda_j)) \cdot 1)\phi(e_j)
\]
\[
= \lim_j \sum_j (\phi(f(\lambda_j) \cdot 1) + \phi(g(\lambda_j) \cdot 1))\phi(e_j)
\]
\[
= \phi(x) + \phi(y)
\]
by the condition (iv), Corollary 1.4(ii) and Lemma 2.1.
**Lemma 3.2.** Let \( u \) and \( v \) be unitaries in \( M \). If \( u \) is selfadjoint, then we have 
\[
\phi(u + v) = \phi(u) + \phi(v).
\]

**Proof.** Put \( e = (1/2)(1 + u) \) and \( w = e + i(1 - e) \) \((i^2 = -1)\). Then \( e \) is a projection in \( M \) and \( w \) is a unitary in \( M \) such that \( w^2 = u \). Since \( w^*vw^* \) is a unitary in \( M \), by the spectral theory, there exists a selfadjoint element \( h \) in \( M \) such that \( w^*vw^* = e^{ih} \).

The map \( f \in \mathcal{C}(\sigma(h)) = \mathcal{C}(\mathbb{R})|\sigma(h) \mapsto f(h) \in C^*(h, 1) \) is a surjective isometric \(*\)-isomorphism from \( \mathcal{C}(\sigma(h)) \) to \( C^*(h, 1) \). So

\[
\left\| e^{ih} - \sum_{k=0}^{n} ((ih)^k/k!) \right\| = \sup \left\{ e^{i\lambda} - \sum_{k=0}^{n} ((i\lambda)^k/k!) : \lambda \in \sigma(h) \right\} \to 0 \quad \text{as} \quad n \to \infty.
\]

Hence \( w^*vw^* \in C^*(h, 1) \subset M \).

Thus it follows that \( \phi(1 + w^*vw^*) = \phi(1) + \phi(w^*vw^*) \) by Lemma 3.1. So we get

\[
\phi(u + v) = \phi(w(1 + w^*vw^*)w) = \phi(w)\phi(1 + w^*vw^*)\phi(w) = \phi(w)(\phi(1) + \phi(w^*vw^*))\phi(w) = \phi(u) + \phi(v) \quad \text{by Lemma 2.5.}
\]

For some pair of projections \( e \) and \( f \) in \( M \), we write \( e \sim f \) (resp. \( e \leq f \)) if there exists a partial isometry \( v \) in \( M \) such that \( vv^* = e \) and \( v^*v = f \) (resp. \( v^*v \leq f \)).

**Lemma 3.3.** Let \( h \) be a nonzero selfadjoint element in \( M \), \( x \) be a nonzero element in \( M \) and \( e \) be a projection in \( M \) such that \( e \leq 1 - e \). Then

\[
\phi(ehe + exe) = \phi(ehe) + \phi(exe).
\]

**Proof.** First of all, we shall note that for any \( x \in M \) with \( \|x\| \leq 1 \), there exists a unitary \( u \) such that \( exe = eue \). In particular, when \( x \) is selfadjoint, \( u \) also

is selfadjoint. In fact, let \( v \) be a partial isometry in \( M \) such that \( vv^* = e \) and \( v^*v \leq 1 - e \). If we put \( u \) to be

\[
y + (e - yy^*)^{1/2}v + v^*(e - y^*y)^{1/2} - vy^*v + g
\]

where \( y = exe \) and \( g = 1 - (e + v^*v) \), \( u \) satisfies all the requirements [2].

If we put \( \gamma(x, y) = \|x\| + \|y\| \), \( h_1 = \gamma(h, x)^{-1}h \) and \( x_1 = \gamma(h, x)^{-1}x \), then there exist unitaries \( u \) and \( v \) in \( M \) such that \( eh_1e = eue \), \( ex_1e = eue \) and \( u \) is selfadjoint. And it follows that

\[
\phi(ehe + exe) = \gamma(h, x)\phi(e(u + v)e) = \gamma(h, x)\phi(e)\phi(u + v)\phi(e) = \gamma(h, x)\phi(e)(\phi(u) + \phi(v))\phi(e) = \phi(ehe) + \phi(exe)
\]

by Lemmas 1.6 and 3.2.

**Lemma 3.4.** Suppose \( e \) is a projection in \( M \) such that \( e \leq 1 - e \). Then \( \phi|eMe \) is additive.

**Proof.** Take arbitrary \( x \) and \( y \) in \( M \) and put \( x = h + ik \) \((i^2 = -1)\) where \( h \) and \( k \) are selfadjoint.
Suppose $h, k$ and $y$ are nonzero. Then

$$
\phi(eze + eye) = \phi(ehe) + \phi(e(ik + y)e)
= \phi(ehe) + \phi(i \cdot 1)(\phi(ke) + \phi(-iye))
= \phi(ehe + e(ik)e) + \phi(eye)
= \phi(eze) + \phi(eye)
$$

by Lemma 3.3. When $h = 0$, $k = 0$ or $y = 0$, the above equalities also hold.

**Lemma 3.5.** Suppose $e$ and $f$ are projections in $M$ such that $e \sim f \leq 1 - e$. Then for any selfadjoint element $h$ in $M$ with $\|h\| \leq 1$, there exists a selfadjoint unitary $u$ in $M$ such that $\{e, h, f\} = \{e, u, f\}$.

**Proof.** Put $u$ to be

$$u = a + a^* + (e - aa^*)^{(1/2)} - (f - a*a)^{(-1/2)} + g$$

where $a = ehf$ and $g = 1 - (e + f)$. Then $u$ satisfies all the requirements.

**Lemma 3.6.** Let $h$ and $k$ be selfadjoint elements in $M$ with $\|h\| \leq 1$ and $\|k\| \leq 1$ and let $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. Suppose $e$ and $f$ are orthogonal equivalent projections in $M$, then we have

$$\phi(\{e, h, f\} + \alpha\{e, k, f\}) = \phi(\{e, h, f\}) + \phi(\alpha\{e, k, f\}).$$

**Proof.** Put $h_1 = \gamma(h, k)^{-1}h$ and $k_1 = \gamma(h, k)^{-1}k$. Then there exist selfadjoint unitaries $u$ and $v$ such that $\{e, h_1, f\} = \{e, u, f\}$ and $\{e, k_1, f\} = \{e, v, f\}$, and it follows that

$$
\phi(\{e, h, f\} + \alpha\{e, k, f\}) = \gamma(h, k)\phi(\{e, u + \alpha v, f\})
= \gamma(h, k)\{\phi(e), \phi(u + \alpha v), \phi(f)\}
= \gamma(h, k)\{\phi(e), \phi(u) + \phi(\alpha v), \phi(f)\}
= \phi(\{e, h, f\}) + \phi(\alpha\{e, k, f\})
$$

by Lemmas 3.5 and 3.2.

Put $x = h_1 + ik_1$, $y = h_2 + ik_2$ ($i^2 = -1$) where $h_j$, $k_j$ ($j = 1, 2$) are selfadjoint elements in $M$. Then Lemma 3.6 leads to the following

**Corollary 3.7.** Let $e$ and $f$ be orthogonal equivalent projections in $M$. Then $\phi|\{e, M, f\}$ is additive where $\{e, M, f\} = \{\{e, x, f\} : x \in M\}$.

**Theorem 3.8.** Let $M$ and $N$ be $AW^*$-algebras and let $\phi$ be a Jordan$^*$-map from $M$ to $N$. Suppose that $M$ has no abelian direct summand and $\phi$ is uniformly continuous on each abelian $C^*$-subalgebra of $M$. Then $\phi$ is additive.

**Proof.** Let $\{p_i\}$ be a family of central orthogonal projections such that $\bigvee_i p_i = 1$ where $M_{p_1}$ has no finite type $I$ direct summand and $M_{p_i}$ ($i \geq 2$) is homogeneous type $I_{n_i}$ for some natural number $n_i$. Then $\phi|_{M_{p_i}}$ is a Jordan$^*$-map from $M_{p_i}$ to $N\phi(p_i)$. We can identify $x$ with $\bigoplus_j x p_j$ ($C^*$-sum) and $\phi(x)$ with $\bigoplus_i \phi(x)\phi(p_i)$. Therefore it is sufficient to prove about $M_{p_i}$ for every $p_i$, and we may assume that
ADDITIVITY OF JORDAN*-MAPS

419

$M$ has an $n \times n$ ($n \geq 2$) matrix unit. Let $\{e_i : i = 1, 2, \ldots, n\}$ be the family of diagonal projections of the matrix unit of $M$. Since $\sum_i \phi(e_i) = 1$, we have

$$
\phi(x) = \left( \sum_i \phi(e_i) \right) \phi(x) \left( \sum_i \phi(e_i) \right) = \sum_i \phi(e_i) \phi(x) \phi(e_i) + 2 \sum_{i<j} \{\phi(e_i), \phi(x), \phi(e_j)\}
$$

$$
= \sum_i \phi(e_i x e_i) + 2 \sum_{i<j} \phi(\{e_i, x, e_j\}).
$$

Since $\phi|e_i Me_i$ and $\phi|\{e_i, M, e_j\}$ are additive, $\phi$ is additive.

The next lemma is due to R. V. Kadison [4]. He proved it in the case of von Neumann algebras. However, his proof holds in the case of AW*-algebras with a slight modification of terminologies.

LEMMA 3.9 [4, THEOREM 10]. Let $M$ (resp. $N$) be an AW*-algebra (resp. $C^*$-algebra) and let $\phi$ be a $C^*$-isomorphism from $M$ to $N$. Then there exists a central projection $f_0$ in $M$ such that $\phi|M f_0$ (resp. $\phi|M (1 - f_0)$) is a *-ring isomorphism (resp. *-ring anti-isomorphism).

THEOREM 3.10. Keep the assumptions on $M$, $N$ and $\phi$ as in Theorem 3.8. There exist four central projections $e_1, e_2, e_3, e_4$ in $M$ such that $\phi|M e_i$ (i = 1, 2, 3, 4) is a linear *-ring isomorphism, $\phi_2$ is a conjugate linear *-ring isomorphism and $\phi_4$ is a conjugate linear anti-isomorphism.

PROOF. By Theorem 3.8 and Lemma 2.4, there exists a unique central projection $e_0$ in $M$ such that $\phi|M e_0$ is a $C^*$-isomorphism of $M e_0$ onto $N e_0$ and $\phi|M (1 - e_0)$ is a conjugate linear map from $M (1 - e_0)$ onto $N (1 - e_0)$ which preserves *-operation and special Jordan product. Put $\psi(x) = \phi(x e_0) + (\phi(x(1 - e_0)))^*$. Then $\psi$ is a $C^*$-isomorphism between $M$ and $N$. So there exists a central projection $f_0$ in $M$ such that $\psi|M f_0$ (resp. $\psi|M (1 - f_0)$) is a linear *-ring isomorphism (resp. linear *-ring anti-isomorphism).

Therefore, we put $e_1 = e_0 f_0$, $e_2 = e_0(1 - f_0)$, $e_3 = (1 - e_0)(1 - f_0)$ and $e_4 = (1 - e_0) f_0$; then $e_1, e_2, e_3$ and $e_4$ satisfy all the requirements.

REMARK. There is an example where the projections $e_1, e_2, e_3$ and $e_4$ in Theorem 3.10 are all nontrivial. In fact, let $M = N = B(H_2) \oplus B(H_2) \oplus B(H_2) \oplus B(H_2)$ where $H_2$ is the 2-dimensional Hilbert space and $x = (x_{ij}) \in B(H_2)$. Let $\phi_1(x) = x$, $\phi_2(x) = t(x_{ij})$ (transpose of $x$), $\phi_3(x) = (x_{ij})$, $\phi(x) = x^*$ and $\phi = \bigoplus_{j=1}^4 \phi_j$. Put $\delta_i = \bigoplus_{j=1}^4 \delta_{ij} \cdot 1$ (i = 1, 2, 3, 4) where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if not (Kronecker's $\delta$). Then $\phi$ is a Jordan*-map from $M$ to $M$, and all $e_i$ (i = 1, 2, 3, 4) are nontrivial, satisfying the requirements of Theorem 3.10.

4. Conjectures.

CONJECTURE 4.1 (S. SAKAI). Theorems 3.8 and 3.10 hold without any hypothesis of continuity.


Finally, the author would like to thank the referee for his comments.
REFERENCES


DEPARTMENT OF BASIC TECHNOLOGY, YAMAGATA UNIVERSITY, YONEZAWA 992, JAPAN