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ON APPROXIMATION BY RATIONALS FROM A HYPERPLANE

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ABSTRACT. Let $E \subset C(K)$ be a subspace of continuous functions defined on a compact Hausdorff space K. We characterize those subspaces of codimension 1 for which the rational functions with denominators and enumerators from E are dense. The condition for the density of this very nonlinear set of functions turns out to be a linear separation condition.

A number of recent discoveries in approximation theory indicated some almost supernatural powers of rational approximation (see [**1**]). That is, starting with a linear subspace $H \subset C([0,1])$, the set of rational functions with the numerator and the denominator in H seems to have density properties disproportionate to its share.

In this note we present the necessary and sufficient condition for the density of such rationals when H is a hyperplane in a $C(X)$-space. Surprisingly enough the condition for the density of this nonlinear set of functions turns out to be a (very linear in spirit) separation condition.

We will need some notation: Let X denote a compact Hausdorff space; $C(X)$ the set of all continuous real-valued functions on X; $\mathcal{M}(X)$ the space of all regular Borel measures on X. The elements from $\mathcal{M}(X)$ are identified with the continuous functionals on $C(X)$. As usual, we define the support $\operatorname{supp} f$ of a continuous function $f \in C(X)$ to be the closure of the set $\{x \in X\colon f(x) \neq 0\}$; the support of a measure μ on X is defined to be the smallest closed set $A \subset X$ such that $\operatorname{supp} f \subset X \backslash A$ implies $\mu(f) = 0$. If U is a subset of X, we let

$$\mathcal{F}(U) = \{f \in C(X)\colon \operatorname{supp} f \subset U,\ f \geq 0\}.$$

For every $\mu \in \mathcal{M}(X)$ we use $\mu = \mu^+ - \mu^-$ to denote the orthogonal decomposition of μ (i.e., $\mu^+ \wedge \mu^- = 0$). Recall that for every pair of measures $\mu, \nu \in \mathcal{M}(X)$ their infimum is given by

$$(\mu \wedge \nu)(f) = \inf\{\mu(g) + \nu(h)\colon 0 \leq g, h,\ g + h = f\},$$

where $f \in C(X)$ is positive (see [**2**, p. 72]). Let $H = \{f \in C(X)\colon \mu(f) = 0\}$ for some $\mu \in \mathcal{M}(X)$. Let

$$R(H) = \{gh^{-1}\colon g, h \in H; h(x) > 0\ \forall\ x \in X\}.$$

LEMMA. *Let $\mu \in \mathcal{M}(X)$ and let U be an open subset of X such that* $\operatorname{supp}\mu^+ \cap U \neq \emptyset \neq \operatorname{supp}\mu^- \cap U$. *Then we can find functions $\varphi_1, \varphi_2 \in \mathcal{F}(U)$ such that $\mu(\varphi_1) < 0 < \mu(\varphi_2)$.*

PROOF. Suppose φ_1 does not exist. Then for all $g \in \mathcal{F}(U)$ we would have $\mu(g) \geq 0$; i.e., $\mu^+(g) \geq \mu^-(g)$. If $f \in \mathcal{F}(U)$, and if $f = g + h$ for positive functions

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g and h, then $g, h \in \mathcal{F}(U)$. Thus $\mu^+(g) + \mu^-(h) \geq \mu^-(g+h) = \mu^-(f)$. Hence the above explicit expression for $\mu^+ \wedge \mu^-$ yields $0 = (\mu^+ \wedge \mu^-)(f) \geq \mu^-(f) \geq 0$ for all $f \in \mathcal{F}(U)$. This would imply $\mu^-(f) = 0$ whenever $\operatorname{supp} f \subset U$; i.e., $\operatorname{supp} \mu^- \subset X \backslash U$, contradicting $\varnothing \neq U \cap \operatorname{supp} \mu^-$.

The proof of the existence of φ_2 is similar. □

THEOREM 1. *Let $1 \in H$. For the set $R(H)$ to be dense in $C(X)$, it is both necessary and sufficient that*

$$\operatorname{supp} \mu^+ \cap \operatorname{supp} \mu^- \neq \varnothing. \tag{1}$$

PROOF. To prove necessity we first assume that $\operatorname{supp} \mu^+ \wedge \operatorname{supp} \mu^- = \varnothing$. Let $f \in C(X)$ be such that $f|_{\operatorname{supp} \mu^+} = 1$, $f|_{\operatorname{supp} \mu^-} - 1$. If $\|f - g/h\| < 1$, then g should be positive on $\operatorname{supp} \mu^+$ and negative on $\operatorname{supp} \mu^-$ (since h is strictly positive). Then $\mu^+(g) > 0$ and $\mu^-(g) < 0$, since μ^+ and μ^- are positive measures, and hence $g \notin H$.

To prove sufficiency, let $x_0 \in \operatorname{supp} \mu^+ \cap \operatorname{supp} \mu^-$, and let f be an arbitrary function from $C(X)$. For $\varepsilon > 0$ let U be a neighborhood of x_0 such that $|(f(x_0) - f(x))| < \varepsilon/2$ for all $x \in U$. By the Lemma, there are functions $\varphi_1, \varphi_2 \in \mathcal{F}(U)$ such that $a = \mu(\varphi_1) > 0$, $b = \mu(\varphi_2) < 0$. We claim that there exists a function $\varphi \in \mathcal{F}(U) \cap H$ with $\varphi(x_0) > 0$. Indeed, pick any $\gamma \in \mathcal{F}(U)$ with $\gamma(x_0) = 1$ and let $r = \mu(\gamma)$. If $r < 0$, then let $\varphi = a(a-r)^{-1} \cdot \gamma - r(a-r)^{-1} \cdot \varphi_1$; if $r > 0$, let $\varphi = r(r-b)^{-1} \cdot \varphi_2 - b(r-b)^{-1} \cdot \gamma$. In any case it follows that $\mu(\varphi) = 0$ and $\varphi \in \mathcal{F}(U)$.

Let $\varphi(x_0) = \rho > 0$. Let $W \subset U$ be a neighborhood of x_0 such that $\varphi_{|W} \geq \rho/2$. Using the same argument as above, we obtain a function $\psi \in \mathcal{F}(W)$ such that $\mu(\psi) = -\mu(f)$; i.e., $f + \psi \in H$. Clearly, for an arbitrary integer N the function

$$r = \frac{f + \psi + N \cdot f(x_0)\varphi}{1 + N\varphi} \in R(H).$$

Pick $N \geq (4\|\psi\| - 2\varepsilon)/(\varepsilon\rho)$. We obtain the inequality $\|f - r\| \leq \varepsilon$, since

(1) For $x \notin U$, we have $|f(x) - r(x)| = 0$.

(2) $x \in U \backslash W$ implies

$$|f(x) - r(x)| = \frac{N\varphi(x)}{1 + N\varphi(x)} |f(x) - f(x_0)| \leq \frac{\varepsilon}{2}.$$

(3) $x \in W$ yields

$$\begin{aligned} |f(x) - r(x)| &\leq \frac{N\varphi(x)}{1 + N\varphi(x)} |f(x) - f(x_0)| + \frac{\|\psi\|}{1 + N\rho/2} \\ &\leq \frac{\varepsilon}{2} + \frac{\|\psi\|}{1 + N\rho/2} \leq \varepsilon. \quad \square \end{aligned}$$

We now obtain two easy generalizations of Theorem 1. In the first we drop the assumptions about the constants.

THEOREM 2. *Let $H = \{f \in C(X) \colon \mu(f) = 0\}$ for some $\mu \in \mathcal{M}(X)$. Then $R(H)$ is dense in $C(X)$ iff condition* (1) *holds.*

PROOF. Condition (1) guarantees the existence of a strictly positive function $\varphi \in H$: Indeed, let $r = \mu(u)$, where u denotes the constant function with value 1. From the Lemma (with $U = X$) we conclude that there is a positive function

φ_1 with $\mu(\varphi_1) = -r$. The function $\varphi = u + \varphi_1$ will be strictly positive, and this function belongs to H. Consider the hyperplane $H_\varphi = \{f/\varphi\colon f \in H\}$. Then

$$H_\varphi = \{f \in C(X)\colon \mu_1(f) = 0\},$$

where $\mu_1 = \varphi\cdot\mu$. It is easy to see that $\mu_1^+ = \varphi\mu^+$, $\mu_1^- = \varphi\cdot\mu^-$ and $1 = \varphi/\varphi \in H_\varphi$. Hence $\operatorname{supp}\mu_1^+ \cap \operatorname{supp}\mu_1^- = \operatorname{supp}\mu^+ \cap \operatorname{supp}\mu^- \neq \varnothing$, and by Theorem 1, $\overline{R(H_\varphi)} = C(X)$. Therefore for every $\varepsilon > 0$ and $f \in C(X)$, there exist $g, h \in H$ such that $\|f - (g/\varphi)/(h/\varphi)\| < \varepsilon$, or, equivalently, $\|f - g/h\| < \varepsilon$. □

Another obvious generalization of Theorem 1 is as follows.

THEOREM 3. *Let M be a finite subset of $\mathcal{M}(X)$ such that for any distinct $\mu, \nu \in M$, $\operatorname{supp}\mu \cap \operatorname{supp}\nu = \varnothing$. Let $H = \{f \in C(X)\colon \mu(f) = 0\ \forall\ \mu \in M\}$. Then $R(H)$ is dense in $C(X)$ iff, for every $\mu \in M$, condition* (1) *holds.* □

We now give some examples for Theorem 2.

EXAMPLES. Consider the subspaces

$$H_1 = H_1(a, b) = \left\{f \in C([0,1])\colon \int_0^a f\,dx = \int_b^1 f\,dx\right\},$$

for some $a, b \in (0, 1)$,

$$H_2 = H_2(\varphi) = \left\{f \in C([0,1])\colon \int_0^1 f\cdot\varphi\,dx = 0\right\};$$

$\varphi \in C\{([0,1])\}$, where $\{\tau\colon \varphi(\tau) = 0\}$ has no interior points,

$$H_3(j) = \operatorname{span}\{\cos k\theta\}_{k=0, k\neq j}^{\infty} \subset C([0,\pi]).$$

From Theorem 2 we have

$$\begin{aligned}\overline{R(H_1)} &= C([0,1]) \quad \text{iff } a = b,\\ \overline{R(H_2)} &= C([0,1]) \quad \text{iff } \varphi \text{ changes sign on } [0,1],\\ \overline{R(H_3)} &= C([0,\pi]) \quad \text{iff } j \neq 0.\end{aligned}$$

The last example follows from the second example and the fact that $H_3(j)$ is dense in the hyperplane

$$H = \left\{f \in C([0,\pi])\colon \int_0^\pi f(\theta)\cos j\theta\,d\theta = 0\right\}.$$

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