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INVARIANT SUBSPACES FOR ALGEBRAS OF SUBNORMAL OPERATORS

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ABSTRACT. Every rationally cyclic subnormal operator has a hyperinvariant subspace.

A bounded linear operator on a Hilbert space is defined to be subnormal if it is the restriction to an invariant subspace of a bounded normal operator. S. W. Brown [**4**] proved that every subnormal operator has a nontrivial invariant subspace. An idea in his proof leads to a stronger theorem, whose proof is easier.

THEOREM. *Let μ be a compactly supported positive measure on the complex plane* $\mathbf{C}$. *Let H be a closed subspace of $L^2(\mu)$ with $1 \in H$. Let A be a subalgebra of $L^\infty(\mu)$ containing the function z such that $AH \subset H$. Then there exists a nontrivial closed subspace K of H such that $AK \subset K$.*

Before proving the theorem, we will point out some consequences. A subnormal operator S on a space H is rationally cyclic if there exists a vector x in H such that the set $\{r(S)x\colon r \in \mathrm{Rat}(\sigma(S))\}$ is dense in H, where $\mathrm{Rat}(\sigma(S))$ denotes the algebra of rational functions with poles off the spectrum of S.

For each such S, there exists a measure μ such that S is unitarily equivalent to multiplication by z on $R^2(\sigma(S), \mu)$, the closure in $L^2(\mu)$ of $\mathrm{Rat}(\sigma(S))$ [**5**, p. 146]. Under this representation each operator that commutes with S is represented by multiplication by a function in $R^2(\sigma(S), \mu) \cap L^\infty(\mu)$, and conversely [**5**, p. 147]. A subspace invariant for every operator that commutes with S is called hyperinvariant. If we let $H = R^2(\sigma(S), \mu)$ and $A = R^2(\sigma(S), \mu) \cap L^\infty(\mu)$ in the theorem, then we obtain a hyperinvariant subspace.

COROLLARY 1. *Every rationally cyclic subnormal operator has a hyperinvariant subspace.*

The following is a trivial consequence of Corollary 1.

COROLLARY 2. *Every subnormal operator S has a subspace invariant for the algebra* $\{r(S)\colon r \in \mathrm{Rat}(\sigma(S))\}$.

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An excellent general reference for subnormal operators is the book by J. B. Conway [**5**]. J. E. Brennan [**1–3**] has worked extensively on rationally invariant subspaces, and the author has found that work instructive.

PROOF OF THE THEOREM. If $H = L^2(\mu)$, then let $K = fH$, where f is a nontrivial characteristic function.

Assume $H \neq L^2(\mu)$. Then there exists a nonzero function g in $L^2(\mu)$ such that $\int fg\,d\mu = 0$ for every f in H. Let $p = 3$ and $q = 3/2$ for the remainder of this paper. By [**1**] and [**5**, p. 316] there exists a point w in $\mathbf{C}$ such that $g(z)(z-w)^{-1}$ is in $L^q(\mu)$, $\mu(\{w\}) = 0$ and

$$\int g(z)(z-w)^{-1}\,d\mu(z) \neq 0.$$

Fix such a point w.

Let $A^p(\mu)$ denote the closure in $L^p(\mu)$ of A. Note that $A^p(\mu) \subset H$. Define a bounded linear functional L on $A^p(\mu)$ by

$$L(f) = \int f(z)g(z)(z-w)^{-1}\,d\mu(z)$$

for f in $A^p(\mu)$. By the Hahn-Banach theorem there exists a norm-preserving extension of L to $L^p(\mu)$. This extension is represented by a function h in $L^q(\mu)$ with $\|h\|_q = \|L\|$. That is,

$$L(f) = \int fh\,d\mu$$

for each f in $A^p(\mu)$. Since the closed unit ball of $A^p(\mu)$ is weakly compact, there exists a function r in $A^p(\mu)$ with $\|r\|_p = 1$ and $L(r) = \|L\|$. Thus,

$$\|h\|_q = \|L\| = L(r) = \int rh\,d\mu \leqslant \|r\|_p \|h\|_q = \|h\|_q.$$

The equality in Holder's inequality above implies that

$$|r|^p = a|h|^q \quad \text{a.e. } (\mu)$$

for some positive constant a, or

$$|r|^2 = b|h| \quad \text{a.e. } (\mu)$$

for some positive constant b.

Let $x = h/r$ on the set where r is nonzero, and zero elsewhere. The function x is in $L^2(\mu)$ because

$$\int |x|^2\,d\mu = \int |h|^2 |r|^{-2}\,d\mu = b^{-1}\int |h|\,d\mu < \infty.$$

Let K be the closure of the linear manifold $(z - w)Ar$. Clearly $AK \subset K$. For each f in A,

$$\int (z-w) frx\, d\mu = \int (z-w) fh\, d\mu$$
$$= \int (z-w) f(z) g(z) (z-w)^{-1}\, d\mu(z)$$
$$= \int fg\, d\mu = 0.$$

But r is in H and

$$\int rx\, d\mu = \int h\, d\mu = \int g(z)(z-w)^{-1}\, d\mu(z) \neq 0.$$

Thus K is a proper subspace of H.

REMARK. In the proof above, if A is the algebra of polynomials (similarly for rational functions), then there exists a constant c such that $f(w) = (fr, c\bar{x})$ for every polynomial f, where (,) denotes the standard inner product on $L^2(\mu)$. The idea of using factorization to obtain point evaluation is due to Brown [**4**]. The fact that there are $L^p(\mu)$-continuous point evaluations is due to Brennan [**1**]. The original contribution of this paper is the method for combining those ideas.

REFERENCES

1. J. E. Brennan, *Invariant subspaces and rational approximation*, J. Funct. Anal. **7** (1971), 285–310.
2. ______, *Invariant subspaces and weighted polynomial approximation*, Ark. Mat. **11** (1973), 167–189.
3. ______, *Point evaluations, invariant subspaces and approximation in the mean by polynomials*, J. Funct. Anal. **34** (1979), 407–420.
4. S. W. Brown, *Some invariant subspaces for subnormal operators*, Integral Equations Operator Theory **1** (1978), 310–333.
5. J. B. Conway, *Subnormal operators*, Pitman, Boston, Mass., 1981.

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