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A GENERALIZATION OF SLOWLY VARYING FUNCTIONS

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ABSTRACT. This note establishes that if the main part of the definition of a slowly varying function is relaxed to the requirement that $\limsup_{x\to\infty}\psi(\lambda x)/\psi(x) < \beta < \infty$ for each $\lambda > 0$, then $\psi(x) = L(x)\theta(x)$, where L is slowly varying and θ is bounded. This is done by obtaining a representation for the function ψ.

1. Introduction. We are concerned with a function ψ which is positive, finite, measurable and defined on $[A, \infty)$ for some $A > 0$, and such that

$$\limsup_{x\to\infty} \frac{\psi(\lambda x)}{\psi(x)} < \beta < \infty \tag{1}$$

for each $\lambda > 0$, where β ($\geqslant 1$) is independent of $\lambda > 0$. We shall call such a function $S - O$ varying. Notice that this definition of an $S - O$ varying function differs from that in [**4**, Appendix]; the reason for the redefinition will be made clear shortly.

Condition (1) generalizes the notion of a slowly varying function [**4**], which requires that $\lim_{x\to\infty}\psi(\lambda x)/\psi(x) = 1$; our development elucidates the structure of $S - O$ varying ψ in relation to the properties of slowly varying functions.

Notice that (1) implies that

$$0 < \frac{1}{\beta} < \liminf_{x\to\infty} \frac{\psi(\lambda x)}{\psi(x)} \tag{2}$$

for each $\lambda > 0$. Passing to the usual additive reformulation, put $f(x) = \log\psi(e^x)$. Then (1) and (2) imply that

$$\forall u \in (-\infty, \infty), \qquad \limsup_{x\to\infty} |f(x+u) - f(x)| \leqslant K < \infty, \tag{3}$$

where K can be taken as $\log\beta$. The main purpose of this note is to prove

THEOREM. *Suppose f is defined, finite and measurable on $[B, \infty)$ for some finite B. Then f satisfies* (3) *for some K if and only if f has the representation*

$$f(x) = a(x) + \int_{n_0}^{x} \varepsilon(u)\,du$$

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for all $x \geqslant n_0$, *where* n_0 ($\geqslant B$) *is some fixed number*, $a(x)$ *is measurable and bounded on* $[n_0, \infty)$, *and* $\varepsilon(x)$ *is measurable on* $[n_0, \infty)$, *with* $\varepsilon(x) \to 0$ *as* $x \to \infty$.

For a slowly varying function in its additive version, K in (3) is zero. A bounded f satisfying (3), where K cannot be taken as zero, is $f(x) = (-1)^{[x]}$. An example of an unbounded f of this kind can be obtained by adding the additive version of any unbounded slowly varying function; e.g., $f(x) = (-1)^{[x]} + \log x$.

To place our result within the context of existing theory of regular variation and its generalizations, we return from the additive formulation to consider all measurable functions ψ which for $x \geqslant A_1$ can be represented in the form

$$\psi(x) = \exp\left\{\alpha(x) + \int_{A_1}^{x} \beta(t)t^{-1}\,dt\right\},$$

where $\alpha(x)$ and $\beta(x)$ are measurable. We then have the following correspondences, where the Theorem corresponds to IV:

I. ψ slowly varying [**4**] $\Leftrightarrow$ $\alpha(x) \to \alpha$, $\beta(x) \to 0$.

II. ψ $R - O$ varying [**4**, Appendix] $\Leftrightarrow$ $\alpha(x)$ bounded, $\beta(x)$ bounded.

III. ψ ER varying [**2**, §2; **3**, (3.8)] $\Leftrightarrow$ $\alpha(x) \to \alpha$, $\beta(x)$ bounded.

IV. ψ $S - O$ varying $\Leftrightarrow$ $\alpha(x)$ bounded, $\beta(x) \to 0$.

It is clear that IV is a subclass of II, the $S - O$ terminology being justified because of the behaviour of $\beta(x)$. The example of an $S - O$ varying function mentioned earlier shows that IV is not a subclass of III. Finally, a positive measurable function ψ is $S - O$ varying on $[A, \infty)$ if and only if it can be written in the form $\psi(x) = L(x)\theta(x)$, $x \geqslant A$, where L is slowly varying at infinity and $\theta(x)$ is such that $\theta(x)$ and $1/\theta(x)$ are bounded on $[A, \infty)$. For further relaxation of condition (1) see [**1**; **2**, Corollary 3.4].

2. Results.

LEMMA. *Under the conditions of the Theorem and integer* U,

$$\sup_{U>0}\ \limsup_{x\to\infty}\left\{\sup_{0\leqslant u\leqslant U} |f(x+u) - f(x)|\right\} < \infty. \tag{4}$$

PROOF. By [**1**, Theorem 1], (3) implies that for any compact interval I in $(-\infty, \infty)$,

$$-\infty < \liminf_{x\to\infty}\left\{\inf_{u\in I}(f(x+u) - f(x))\right\}, \quad \limsup_{x\to\infty}\left\{\sup_{u\in I}(f(x+u) - f(x))\right\} < \infty,$$

whence

$$\limsup_{x\to\infty}\left\{\sup_{u\in I}|f(x+u) - f(x)|\right\} < \infty. \tag{5}$$

Now let $C/2$ exceed both $\sup_{u\geqslant 0}\limsup_{x\to\infty}|f(x+u) - f(x)|$ (from (3)), and $\limsup_{x\to\infty}\sup_{0\leqslant u\leqslant 1}|f(x+u) - f(x)|$ (from (5)). Then for any positive integer U there exists $X = X(U) > 0$ such that for all $x \geqslant X$,

$$|f(x+k) - f(x)| < C/2 \qquad (k = 1, 2, \ldots, U),$$

$$\sup_{0\leqslant u\leqslant 1}|f(x+u) - f(x)| < C/2.$$

Then for $0 \leqslant u \leqslant U$ and $x \geqslant X$,

$$|f(x+u)-f(x)| \leqslant |f(x+[u])-f(x)| + |f(x+u)-f(x+[u])| < C. \quad \square$$

PROOF OF THE THEOREM. Define $n_0 < n_1 < n_2 < \cdots$ as follows. Let $n_0 \geqslant X(1)$ be an integer. Having defined n_{k-1}, (4) allows us to find a positive integer m_k such that, on setting $n_k = n_{k-1} + k(m_k + 1)$,

$$\sup_{x \geqslant n_k} \sup_{0 \leqslant u \leqslant k+1} |f(x+u) - f(x)| < C. \tag{6}$$

Now for each $k = 1, 2, \ldots$ and $m = 0, 1, \ldots, m_k$ define

$$\varepsilon(x) = \left(f(n_{k-1} + mk + k) - f(n_{k-1} + mk)\right)/k,$$
$$n_{k-1} + mk \leqslant x < n_{k-1} + mk + k.$$

On this interval $|\varepsilon(x)| \leqslant C/k$ so $\varepsilon(x) \to 0$ as $x \to \infty$.

Now, given any $x \geqslant n_0$, we may write it as $x = n_{k-1} + mk + y$ for some $k \in \{1, 2, \ldots\}$, $m \in \{0, \ldots, m_k\}$ and $0 \leqslant y < k$, so that

$$\begin{aligned} f(x) - f(n_0) - \int_{n_0}^{x} \varepsilon(u)\,du &= f(x) - f(n_{k-1} + mk) - \int_{n_{k-1}+mk}^{x} \varepsilon(u)\,du \\ &= f(n_{k-1} + mk + y) - f(n_{k-1} + mk) \\ &\quad - \frac{y}{k}\left(f(n_{k-1} + mk + k) - f(n_{k-1} + m_k)\right), \end{aligned}$$

which by (6) is in absolute value $< 2C$. $\square$

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