

## UNIMODULAR MATRICES IN BANACH ALGEBRA THEORY

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**ABSTRACT.** Let  $A$  be a ring with 1 and denote by  $L$  (resp.  $R$ ) the set of left (resp. right) invertible elements of  $A$ . If  $A$  has an involution  $*$ , there is a natural bijection between  $L$  and  $R$ . In general, it seems that there is no such bijection; if  $A$  is a Banach algebra,  $L$  and  $R$  are open subsets of  $A$ , and they have the same cardinality. More generally, we prove that the spaces  $U_k(A^n)$  of  $n \times k$ -left-invertible matrices and  ${}_kU(A^n)$  of  $k \times n$ -right-invertible matrices are homotopically equivalent. As a corollary, we answer negatively two questions of Rieffel [12].

1. Let  $A$  be a ring (with an identity element 1) and  $k \leq n$ . A matrix  $x \in A^{n \times k}$  is *unimodular* if there exists  $y \in A^{k \times n}$  such that  $y \cdot x = 1 \in M_k(A) = A^{k \times k}$ . We denote by  $U_k(A^n)$  the set of all unimodular matrices of  $A^{n \times k}$ , and  $S_k(A^n) = \{(x, y) \in A^{n \times k} \times A^{k \times n}: y \cdot x = 1\}$ . The first projection induces a mapping  $p: S_k(A^n) \rightarrow U_k(A^n)$ .

1.1. **PROPOSITION.** Let  $A$  be a ring,  $x \in A^{n \times k}$ , and define  $T_x: M_n(A) \rightarrow A^{n \times k}$  by  $T_x(\sigma) = \sigma x$ . Then  $x$  is unimodular if and only if  $T_x$  is a direct epimorphism; i.e., there is an  $A$ -linear map  $S: A^{n \times k} \rightarrow M_n(A)$  such that  $T_x \circ S = 1_{A^{n \times k}}$ .

**PROOF.** Obvious.

From now on,  $A$  denotes a (real or complex) Banach algebra. Our purpose is to study the fibration properties of the mappings

$$t_x = T_x|_{\text{GL}_n(A)}: \text{GL}_n(A) \rightarrow U_k(A^n),$$

$$\text{for } x \in U_k(A^n) \text{ and } p: S_k(A^n) \rightarrow U_k(A^n).$$

1.2. **LEMMA.** (i)  $U_k(A^n)$  is open in  $A^{n \times k}$ .

(ii)  $t_x$  is continuous, for every  $x \in U_k(A^n)$ .

(iii)  $p$  is continuous.

**PROOF.** It is clear that  $t_x$  and  $p$  are continuous, because they are restrictions of linear mappings. If  $x_0 \in A^{n \times k}$  is unimodular and  $y \cdot x_0 = 1 \in M_k(A)$ , there is a neighborhood  $U$  of  $x_0$  in  $A^{n \times k}$  such that  $y \cdot x \in \text{GL}_k(A)$  for every  $x \in U$ , for the set of invertible elements of a Banach algebra is open. But it is evident that  $U$  is contained in  $U_k(A^n)$ . Thus  $U_k(A^n)$  is open as claimed.

In view of the theory of Banach manifolds, as developed in Lang [7], we can prove the next result. Recall that a differentiable map  $f: X \rightarrow Y$  is a *submersion* at  $x \in X$  if the derivative of  $f$  at  $x \in X$  is a direct epimorphism from the tangent space of  $X$  at  $x$  onto the tangent space of  $Y$  at  $f(x)$ .

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1.3. PROPOSITION. *Let  $A$  be a Banach algebra and  $x \in A^{n \times k}$ . Then the following hold.*

- (1) *If  $x \in U_k(A^n)$ , then  $t_x: \text{GL}_n(A) \rightarrow U_k(A^n)$  is a submersion.*
- (2)  *$S_k(A^n)$  is a Banach manifold.*

PROOF. (1) If  $x \in U_k(A^n)$ , the derivative of  $t_x$  at any  $\sigma \in \text{GL}_n(A)$  coincides with  $T_x$ , which is a direct epimorphism, by 1.1, so  $t_x$  is a submersion at  $\sigma$ .

(2) Following Raeburn [11], we need only prove that the derivative of  $\phi: A^{n \times k} \times A^{k \times n} \rightarrow M_k(A)$ , defined by  $\phi(x, y) = y \cdot x$ , at any  $(x_0, y_0) \in S_k(A^n)$  is a direct epimorphism for  $S_k(A^n) = \phi^{-1}(1)$ .

Now, the derivative of  $\phi$  at  $(x_0, y_0)$  is the  $A$ -linear map  $(w, z) \rightarrow z \cdot x_0 + y_0 \cdot w$  ( $w \in A^{n \times k}$ ,  $z \in Z^{k \times n}$ ), and the map  $a \rightarrow 1/2(x_0 a, a y_0)$  ( $a \in M_k(A)$ ) is a right inverse of  $D\phi(x_0, y_0)$ .

1.4. COROLLARY. *For every  $x \in U_k(A^n)$ , the map  $t_x: \text{GL}_n(A) \rightarrow U_k(A^n)$  is open, and its image  $\text{GL}_n(A) \cdot x$  is open and closed in  $U_k(A^n)$ .*

PROOF. Clearly a submersion is an open map. Thus, for every  $z \in U_k(A^n)$ ,  $\text{GL}_n(A) \cdot z$  is open in  $U_k(A^n)$ . But  $\text{GL}_n(A) \cdot z$  is the orbit of  $z$  under the action of  $\text{GL}_n(A)$  on  $U_k(A^n)$  given by left multiplication, so it is closed in  $U_k(A^n)$ : in fact, when all orbits are open, they are also closed.

1.5. REMARKS. (1) Observe that, if  $z \in U_k(A^n)$  belongs to the orbit  $\text{GL}_n(A) \cdot x$ , the entire connected component of  $z$  in  $U_k(A^n)$  is contained in the same orbit. This obvious (in view of 1.4) fact has interesting consequences, as we shall see.

(2) If in the proof of 1.4 we consider the action of  $\text{GL}_n(A)_0$ , the connected component of the neutral element, on  $U_k(A^n)$ , we get that the connected component of  $x$  in  $U_k(A^n)$  is  $\text{GL}_n(A)_0 \cdot x$ .

1.6. THEOREM. *Let  $A$  be a Banach algebra and  $x \in U_k(A^n)$ . Then the following hold.*

- (i)  *$t_x$  induces a principal locally trivial fibre bundle  $\text{GL}_n(A) \rightarrow \text{GL}_n(A) \cdot x$ .*
- (ii)  *$t_x: \text{GL}_n(A) \rightarrow U_k(A^n)$  is a Serre fibration.*
- (iii)  *$t_x$  is surjective if and only if the induced mapping  $\pi_0(t_x): \pi_0(\text{GL}_n(A)) \rightarrow \pi_0(U_k(A^n))$  is onto. In particular,  $t_x$  is surjective if  $U_k(A^n)$  is connected.*

PROOF. Let  $G_x$  be the stabilizer of  $x$ ; i.e.,  $G_x = \{\sigma \in \text{GL}_n(A): \sigma \cdot x = x\}$ . Then  $G_x$  is a Banach-Lie subgroup of  $\text{GL}_n(A)$ , and, by 1.4,  $\text{GL}_n(A) \cdot x$  may be identified with the homogeneous space  $\text{GL}_n(A)/G_x$ . So, assertion (i) follows from the general theory of Banach-Lie groups as Bourbaki [1] or Raeburn [11].

For the proof of (ii), it is well known that any locally trivial bundle over a paracompact base space is a Serre fibration (see Dold [3]). In our case,  $U_k(A^n)$  is metrizable, a fortiori paracompact. The proof of (iii) is now obvious.

The problem of determining whether an element  $z$  of  $U_k(A^n)$  belongs to  $\text{GL}_n(A^n) \cdot x$  for a fixed  $x \in U_k(A^n)$  is, in general, very difficult. Let us give an example. Let  $A$  be the algebra of real continuous functions on the unit sphere  $S^{n-1}$  of  $R^n$ , and let  $x_i$ ,  $i = 1, \dots, n$ , be the coordinate functions. Clearly,  $x = (x_1, \dots, x_n)$  belongs to

$U_1(A^n)$  for  $x \cdot x = \sum_{i=1}^n x_i^2 = 1$ . Let  $t$  be the map  $\sigma \rightarrow \sigma \cdot e_1$  (where  $e_1$  is the first canonical vector of  $A^n$ ) from  $GL_n(A)$  into  $U_1(A^n)$ . We shall show that  $x$  belongs to  $GL_n(A) \cdot e_1$  only if  $n = 2, 4$ , or  $8$ . In fact,  $x \in GL_n(A) \cdot e_1$  means that there is a basis  $\{v_1, v_2, \dots, v_n\}$  of  $A^n$  with  $v_1 = x$ . Let  $S$  be the free supplement of  $A \cdot x$  generated by  $\{v_2, \dots, v_n\}$  and  $S' = \{v \in A^n: \langle v, x \rangle = 0\}$ . Then  $S'$  is another supplement of  $A \cdot x$ , so  $S$  and  $S'$  are isomorphic and  $S'$  must be free. But  $S'$  identifies with the  $A$ -module of continuous sections of the tangent bundle  $\tau(S^{n-1})$  of  $S^{n-1}$  (see Swan [13]). Thus, since  $S'$  is free,  $\tau(S^{n-1})$  must be a trivial bundle, and this may happen only if  $n = 2, 4$ , or  $8$ , as claimed.

Let  $A$  be a ring, and let  $e$  be the  $n \times k$ -matrix whose columns are, in order, the first  $k$  canonical vectors of  $A^n$ . Let us set  $t = t_e: GL_n(A) \rightarrow U_k(A^n)$ .  $A$  is  $(n, k)$ -Hermite if  $t$  is onto. (This terminology is consistent with that of Lam [6], at least for commutative rings.) The following lemma shows that the choice of  $e$  is irrelevant.

1.7. LEMMA. *Let  $A$  be a ring. Then  $A$  is  $(n, k)$ -Hermite if and only if  $t_x: GL_n(A) \rightarrow U_k(A^n)$  is onto for every  $x \in U_k(A^n)$ .*

PROOF. Suppose that  $A$  is  $(n, k)$ -Hermite, and take  $z$  and  $x$  in  $U_k(A^n)$ . Then there exist  $\tau$  and  $\sigma$  in  $GL_n(A)$  such that  $z = \tau \cdot e$  and  $x = \sigma \cdot e$ ; then  $t_x(\tau\sigma^{-1}) = \tau \cdot \sigma^{-1} \cdot x = z$ . The converse is trivial.

From 1.6(iii) we get

1.8. PROPOSITION. *Let  $A$  be a Banach algebra. Then  $A$  is  $(n, k)$ -Hermite if and only if  $\pi_0(t)$  is onto. In particular,  $A$  is  $(n, k)$ -Hermite if  $U_k(A^n)$  is connected.*

1.9. EXAMPLES. (1) Let  $A$  be a complex commutative Banach algebra with spectrum  $X(A)$ . It is well known that  $\pi_0(U_k(A^n))$  coincides with the set of homotopy classes of maps from  $X(A)$  into  $U_k(\mathbb{C}^n)$  (see Novodvorski [9], Taylor [14] and Raeburn [11]). Lin [8] proved that  $A$  is  $(n, k)$ -Hermite if and only if  $C(X(A))$ , the algebra of complex continuous functions on  $X(A)$ , is; moreover, for  $x$  and  $z$  in  $U_k(A^n)$ ,  $z$  belongs to  $GL_n(A) \cdot x$  if and only if the Gelfand transform  $\hat{z}$  belongs to  $GL_n(C(X(A))) \cdot \hat{x}$ . In particular, if  $X(A)$  is dominated by an  $m$ -dimensional space, with  $m \leq 2(n - k)$ , then  $A$  is  $(n, k)$ -Hermite. This last fact has also been proved by Lin [8].

(2) Let  $\mathcal{H}$  be a Hilbert space and  $A$  the algebra of bounded linear operators on  $\mathcal{H}$ . Then  $GL_n(A)$  is contractible for every  $n$  (Kuiper [5]). Now,  $A$  is isomorphic to  $A^{n \times k}$  (as  $A$ -modules and as Banach spaces), and it is easy to see that  $U_k(A^n)$  is homeomorphic to  $U_1(A^1)$ , which is disconnected, as we shall see 2.6(iii). Then  $A$  cannot be  $(n, k)$ -Hermite for any  $k \leq n$ .

(3) If the stable range of  $A$  is  $\leq n - k$ ,  $A$  is  $(n, k)$ -Hermite (see Corach and Larotonda [21]).

2. In this section we study the fibration properties of the map  $p: S_k(A^n) \rightarrow U_k(A^n)$  when  $A$  is a Banach algebra.

2.1. THEOREM. *Let  $A$  be a Banach algebra. Then the projection onto the first coordinate  $p: S_k(A^n) \rightarrow U_k(A^n)$  is a locally trivial bundle with contractible fibre.*

PROOF. For  $(x_0, y_0) \in S_k(A^n)$  we define  $\phi: A^{n \times k} \rightarrow \text{GL}_n(A) \cdot x_0 \subset A^{n \times k}$  by  $\phi(z) = \exp(x, y_0) \cdot x_0$ . The derivative of  $\phi$  at  $0 \in A^{n \times k}$  is the identity map of  $A^{n \times k}$ , so by the inverse function theorem [1, 7] there exist neighborhoods  $U$  of 0 in  $A^{n \times k}$  and  $V$  of  $x_0$  in  $U_k(A^n)$  such that  $\phi$  is a diffeomorphism from  $U$  onto  $V$ . We define a section of  $t_{x_0}$  over  $V$ ,  $s: V \rightarrow \text{GL}_n(A)$ , by  $s(x) = \exp(\phi^{-1}(x) \cdot y_0)$ : In fact,  $t_x \circ s(x) = s(x) \cdot x_0 = x$  for every  $x \in V$  and  $s(x_0) = 1_{M_n(A)}$ .

We now define a trivializing map  $g: p^{-1}(V) \rightarrow V \times N$  by  $g(x, y) = (x, y \cdot s(x) - y_0)$ , where  $N = \{y \in A^{k \times n}: y \cdot x_0 = 0\}$ ; the inverse of  $g$  is  $h: V \times N \rightarrow p^{-1}(V)$ ,  $h(x, z) = (x, (z + y_0) \cdot s(x)^{-1})$ . The fibre  $p^{-1}(x)$ , for  $x \in V$ , is the affine manifold  $\{(x, (y + y_0) \cdot s(x)^{-1}): y \in N\}$ , obviously contractible.

2.2. COROLLARY. *The projection  $p$  is a homotopy equivalence and it admits a global section.*

PROOF. From the examination of the homotopy sequence of  $p$ , it follows that  $p$  is a weak homotopy equivalence, so, by a result of Palais [10, Theorem 15], since  $S_k(A^n)$  and  $U_k(A^n)$  are metrizable manifolds (by 1.2 and 1.3),  $p$  is a homotopy equivalence. The second assertion follows from a result of Godement (see Dold [3, p. 223]) which states that a fibre bundle with contractible fibre over a paracompact space has a global section.

2.3. COROLLARY. *For every compact pair  $(Z, Y)$ , every map  $f: (Z, Y) \rightarrow (U_k(A^n), x_0)$ , and every  $y_0 \in A^{k \times n}$  such that  $y_0 \cdot x_0 = 1 \in M_k(A)$ , there is a lifting  $\tilde{f}: (Z, Y) \rightarrow (S_k(A^n), (x_0, y_0))$ ; i.e.,  $p\tilde{f} = f$ .*

2.4. COROLLARY. *For every compact space  $X$  the natural inclusion  $U_k(A(X)^n) \rightarrow C(X, U_k(A^n))$  (where  $A(X)$  is the algebra of maps from  $X$  into  $A$ ) is a homeomorphism.*

We have been studying left unimodular matrices. We can also consider right unimodular matrices:  $y \in A^{k \times n}$  ( $k \leq n$ ) is *right unimodular* if there exists  $x \in A^{n \times k}$  such that  $y \cdot x = 1 \in M_k(A)$ . Of course, the set  ${}_k U(A^n)$  of right unimodular matrices is the image of the second projection  $p_2: S_k(A^n) \rightarrow A^{k \times n}$ . We may reproduce all the results obtained for  $U_k(A^n)$  in this new context. In particular, we may prove that  $p_2: S_k(A^n) \rightarrow {}_k U(A^n)$  is a homotopy equivalence. Thus, we have

2.5. COROLLARY.  *$U_k(A^n)$  and  ${}_k U(A^n)$  are homotopically equivalent.*

REMARK. This result answers negatively question 4.8 of Rieffel [12] and generalizes his Proposition 10.2 to any Banach algebra  $A$ .

We now consider the case  $n = k = 1$ , so  $U_1(A^1) = L$  and  ${}_1 U(A^1) = R$ .

2.6. THEOREM. *Let  $A$  be a Banach algebra. Then the following hold.*

(i)  *$L$  and  $R$  are homotopically equivalent.*

(ii) *The connected components of 1 in  $L$ ,  $R$  and  $G$  (= group of units of  $A$ ) coincide; in particular,  $G$  is connected if  $L$  or  $R$  are connected and, in this case,  $G = L = R$ . Thus,  $L$  and  $R$  are disconnected if  $G \neq L$  or  $G \neq R$ .*

The proof follows directly from 2.5 and the following obvious lemma.

2.7. LEMMA.  $G$  is open and closed in  $L$  and  $R$ .

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