DECOMPOSITION OF POSITIVE PROJECTIONS ONTO JORDAN ALGEBRAS

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Abstract. If a positive projection $P$ from a $C^*$-algebra onto a Jordan algebra can be decomposed as a sum of maps satisfying certain inequalities of the Schwarz type, then $P$ is actually a sum of completely positive and completely copositive maps.

Suppose that $P$ is a positive unital projection on a $C^*$-algebra $A$ such that $P(A)_{sa}$ is a $JC$-algebra. $P$ is said to be decomposable if it is the sum of completely positive and completely copositive maps on $A$. It was shown in [6] that $P$ is decomposable if and only if $P(A)_{sa}$ is reversible in the sense that it is closed under symmetric products of the type $x_1x_2\cdots x_n + x_n\cdots x_2x_1$. Using this result, it was shown in [5] that $P$ is decomposable whenever it can be expressed as a sum of 2-positive and 2-copositive maps. We now give an improvement to the latter result, using a technique of [2].

If $A$ is a unital $C^*$-algebra and $H$ is a Hilbert space, then a linear map $\varphi: A \to B(H)$ is called a Schwarz map [4] if it satisfies the inequality $\varphi(a)^*\varphi(a) \leq \varphi(a^*a)$ for all $a \in A$. Similarly $\varphi$ is an anti-Schwarz map if it satisfies $\varphi(a)\varphi(a)^* \leq \varphi(a^*a)$ for all $a \in A$. Every unital 2-positive map $\varphi$ is a Schwarz map, but not conversely [1]. Similarly every unital 2-copositive map is an anti-Schwarz map. It follows from [1, Lemma 2.1] that a unital linear map $\varphi$ is a Schwarz map if and only if

$$\varphi \otimes I_2\left(1 \otimes e_{11} + a \otimes e_{12} + a^* \otimes e_{21} + a^*a \otimes e_{22}\right) \geq 0,$$

for all $a \in A$. There is obviously a similar condition for anti-Schwarz maps.

In order to study the decomposition of $P$ we need to consider nonunital maps $\varphi$. The ideas of [2, Lemma 3] are useful here. Let $h = \varphi(1)^{1/2}$, with support projection $p$. Then $h^{-1}$ exists as a positive (unbounded) selfadjoint operator affiliated with $pB(H)p$, and we can define a positive unital linear map $\tilde{\varphi}$ on $A$ by $\tilde{\varphi}(a) = h^{-1}\varphi(a)h^{-1}$. We shall say that $\varphi$ is sesqui-positive if $\tilde{\varphi}$ is a Schwarz map and that $\varphi$ is sesqui-copositive if $\tilde{\varphi}$ is an anti-Schwarz map. We then have the following characterization of sesqui-positive maps, which has an obvious analogue for sesqui-copositive maps.

Lemma 1. The following statements are equivalent, for a positive linear map $\varphi: A \to B(H)$.

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(1) $\phi$ is sesqui-positive.
(2) $\phi(a^*a) \geq \phi(a)^*\phi(1)^{-1}\phi(a)$, for all $a \in A$.
(3) $\phi \otimes I_2(1 \otimes e_{11} + a \otimes e_{12} + a^* \otimes e_{21} + a^*a \otimes e_{22}) \geq 0$, for all $a \in A$.

**Proof.** The right-hand side of the inequality in (2) is to be interpreted as the strong limit of the sequence $\phi(a)^*(\phi(1) + 1/n)^{-1}\phi(a)$ as $n \to \infty$, as in [3]. The equivalence of the three conditions follows from [2, Lemma 3.1, Lemma 2.1]. □

We need two more preliminary lemmas before proving the main result.

**Lemma 2.** If $\phi: A \to B(H)$ is a sesqui-positive (sesqui-copositive) linear map, then the same is true for the second dual map $\phi^{**}$.

**Proof.** This follows easily from condition (3) of Lemma 1, by approximating elements of $A^{**}$ in the $\sigma$-strong* topology. □

**Lemma 3.** Let $M$ be a von Neumann algebra, $B$ a JW-subalgebra of $M_{sa}$ and $\phi$ a positive linear map on $M$ such that $\phi(x) \leq x$ whenever $0 \leq x \in B$. Then $\phi(b) = \phi(1)b = b\phi(1)$ for all $b \in B$.

**Proof.** Given a projection $p \in B$, we have $0 \leq \phi(p) \leq p$, so that $(1 - p)\phi(p) = 0$. Replacing $p$ by $1 - p$ gives $p\phi(1 - p) = 0$, and subtraction of these two equations results in $\phi(p) = p\phi(1)$. The result follows, since $B$ is the closed linear span of its projections. □

**Theorem 4.** Let $A$ be a unital $C^*$-algebra and $P$ a positive unital projection on $A$ such that $B = P(A)_{sa}$ is a JC-subalgebra of $A$. If $P = \phi + \psi$, where $\phi$ is sesqui-positive and $\psi$ is sesqui-copositive, with $\phi(1)$, $\psi(1)$ invertible, then $P$ is decomposable.

**Proof.** It is enough to show that the second dual map $P^{**}$ is decomposable. We may identify $B^{**}$ with the ultraweak closure of $B$ in the von Neumann algebra $A^{**}$, so that $B^{**} = P^{**}(A^{**})$ is a JW-subalgebra of $A$ [7, Theorem 1]. Therefore, by Lemma 2, we may suppose that $A$ is a von Neumann algebra, $B$ is a JW-subalgebra and $P$ is a normal projection satisfying $P(A)_{sa} = B$. To show that $P$ is decomposable, it suffices to show that $B$ is reversible [6, Corollary 7.4].

Let $h = \phi(1)^{1/2}$ and $k = \psi(1)^{1/2}$. Define $\tilde{\phi}$ and $\tilde{\psi}$ to be the positive unital linear maps associated with $\phi$ and $\psi$ respectively. Then $\tilde{\phi}$ is a Schwarz map and $\tilde{\psi}$ is an anti-Schwarz map, and

$$P(a) = h\tilde{\phi}(a)h + k\tilde{\psi}(a)k.$$ 

Note that $h^2 + k^2 = 1$, from which it follows that $h$ and $k$ commute.

Now if $b \in B$ then $P(b) = b$, so by Lemma 3, $\phi(b) = h^2b = bh^2$ and $\psi(b) = k^2b = bk^2$. In particular $hb = bh$ and $kb = bk$, whenever $b \in B$, so that $\tilde{\phi}(b) = \tilde{\psi}(b) = b$. Since $B$ is a Jordan algebra, we also have $\tilde{\phi}(b^2) = \tilde{\psi}(b^2) = b^2$.

We can now show that $B$ is reversible. We may suppose that $A$ is the von Neumann algebra generated by $B$. It follows from [4, Theorem] and its anti-Schwarz analogue that $\tilde{\phi}$ is the identity map and $\tilde{\psi}$ is an anti-automorphism of order 2 of $A$. 
which fixes the elements of $B$. Also, $h$, $k$ belong to the centre of $A$ and
\[ P(a) = h^2\phi(a) + k^2\psi(a). \]

It is clear from this that $B$ is reversible, which proves the result. \(\square\)

**Remarks.**

1. The argument above proves more than stated. In fact if $A$ is the von Neumann algebra generated by a $JW$-algebra $B$ and $P$ is a positive unital projection satisfying $P(A_\infty) = B$ and $P = \phi + \psi$, with $\phi$, $\psi$ as before, then
\[ P(a) = z_1a + z_2\alpha(a), \]
where $z_1, z_2 \geq 0$ are in the centre of $A$, $z_1 + z_2 = 1$ and $\alpha$ is a * antiautomorphism of order 2 of $A$.

2. If $B$ is a spin factor of dimension other than 3, 4 or 6, then the corresponding projection $P$ fails to have a decomposition as in [6, Theorem 4].

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**Note added in proof.** E. Størmer has kindly drawn the author's attention to a paper of T. B. Andersen, *On multipliers and order-bounded operators in $C^*$-algebras*, Proc. Amer. Math. Soc. 25 (1970), 896–899 which proves a $C^*$-algebraic version of Lemma 3.

**References**


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