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A NECESSARY CONDITION FOR L^2 STABILITY OF QUASILINEAR CONSERVATION LAWS

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ABSTRACT. This paper proves: It is the necessary condition for L^2 stability of quasilinear conservation laws that the solution is absolutely continuous.

Consider quasilinear conservation laws

$$u_t + f(u)_x = 0, \qquad -\infty < x < +\infty, \tag{1}$$

where $u(x,t) \in \mathbf{R}^n$ and $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is a smooth nonlinear function.

Function $u(x,t) \in L^\infty(\Omega) \cap BV(\Omega)$, $\Omega \in \mathbf{R}^2$ is called a weak solution if u satisfies (1) in the sense of distributions. Here $BV(\Omega)$ denotes the space of functions whose first-order derivatives are locally finite Borel measures [**7**].

It is well known that in general the initial value problem to (1) does not have a globally defined smooth solution; on the other hand, the uniqueness is lost in the broader class of above weak solutions. In order to single out an admissible solution, some criterion is needed. A number of such criteria have been proposed, e.g., Lax's [**4**], Liu's [**5**], viscosity [**6**], entropy [**3**] and entropy rate [**1**] criteria, which are all equivalent when applied to weak shocks with genuinely nonlinear fields.

In [**2**], R. J. DiPerna discussed the uniqueness of solutions to (1) under the entropy criterion.

A function $\eta\colon \mathscr{D} \to \mathbf{R}$ defined on an open domain $\mathscr{D} \subset \mathbf{R}^n$ is called an entropy for (1) with entropy flux $q\colon \mathscr{D} \to \mathbf{R}$, if an additional conservation law

$$\eta(u)_t + q(u)_x = 0 \tag{2}$$

holds for all smooth solutions of (1). According to the entropy criterion a weak solution $u(x,t)$ with range in $\mathscr{D}$ is admissible if it satisfies

$$\eta(u)_t + q(u)_x \leqslant 0 \tag{3}$$

in the sense of distributions. Under the entropy criterion, the establishing of uniqueness and stability of solutions to (1) generally depends on some L^2 inequalities [**2**].

As to the L^2 stability of (1), R. J. DiPerna proved the following theorem [**2**]:

THEOREM 1 (DIPERNA). *Suppose that $w(x,t)$ is a Lipschitz continuous solution and $u(x,t)$ is an admissible weak solution of* (1) *in $\mathscr{S}(T) = \{(x,t)\colon 0 \leqslant t < T\}$. Then*

$$\int_{|x|\leqslant M} |u(x,t) - w(x,t)|^2\,dx \leqslant C_2 \int_{|x|\leqslant M + C_1 t} |u(x,0) - w(x,0)|^2\,dx, \tag{4}$$

where the constant C_1 depends on f and L^∞-norms of w and u while the constant C_2

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depends on f, T, the L^∞-norms of w and u and the Lipschitz constant of $w(x,0)$.

The above theorem says that the Lipschitz continuous solution of (1) is L^2 stable in the class of admissible weak solutions.

In this paper, we prove conversely that if the solution of (1) is L^2 stable, then it must be absolutely continuous. Therefore, it is impossible to establish L^2 stability for any discontinuous solutions of (1).

Our main result is

THEOREM 2. *Suppose the weak solution $u(x,t)$ of (1) is L^2 stable and $u_0(x) \equiv u(x,0)$ is locally Lipschitz continuous. Then u is absolutely continuous.*

PROOF. Clearly, $u(x+\varepsilon, t)$ is also a solution of (1) for $\varepsilon > 0$. By (4) we have

$$\int_{|x|\leqslant M} |u(x+\varepsilon,t) - u(x,t)|^2\,dx \leqslant C_2 \int_{|x|\leqslant M + C_1 t} |u_0(x+\varepsilon) - u_0(x)|^2\,dx$$

and hence

$$\int_{|x|\leqslant M} \frac{|u(x+\varepsilon,t) - u(x,t)|^2}{\varepsilon^2}\,dx \leqslant C_2 \int_{|x|\leqslant M + C_1 t} \frac{|u_0(x+\varepsilon) - u_0(x)|^2}{\varepsilon^2}\,dx$$

$$\leqslant C_2(M + C_1 T)(\operatorname{Lip} u_0)^2 = \text{const}$$

where $\operatorname{Lip} u_0$ is the Lipschitz constant of $u_0(x)$ for $|x| \leqslant M + C_1 T$. In view of the weak compactness of Banach space L^2, letting $\varepsilon \to 0^+$, we get

$$\int_{|x|\leqslant M} |u_x|^2\,dx \leqslant \text{const}.$$

By using the Cauchy inequality we obtain

$$\int_{|x|\leqslant M} |u_x|\,dx \leqslant \text{const}. \tag{5}$$

The estimate (5) implies that $u(x,t)$ is an absolutely continuous solution. This completes the proof.

We note that in our theorem the system (1) may be neither genuinely nonlinear nor strictly hyperbolic.

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