SETS OF p-POWERS AS IRREDUCIBLE CHARACTER DEGREES

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Abstract. In this paper it is shown that every finite set of powers of the prime $p$ which contains $p^0 = 1$ occurs as the full set of degrees of the irreducible characters of some $p$-group.

1. Let $G$ be a finite group and write $\text{c.d.}(G)$ to denote the set of numbers which occur as degrees of irreducible complex characters of $G$. There has been occasional interest in the question of which finite sets of positive integers can occur as $\text{c.d.}(G)$. An obvious necessary condition is that 1 must lie in the set. This is not, however, sufficient since, for instance, the set $\{1, 2, 3, 5\}$ is not $\text{c.d.}(G)$ for any $G$. (This follows from Theorem 6.1 of [1] where it is shown that if $\text{c.d.}(G)$ consists of just 1 and prime numbers, then at most two primes occur.)

The purpose of this note is to construct examples which show that for each prime $p$, every finite set of powers of $p$ which contains 1 occurs as $\text{c.d.}(G)$ for some $p$-group $G$ with nilpotence class $\leq 2$.

Some sets of powers of $p$, such as $\{1, p\}$, can occur as $\text{c.d.}(G)$ for $p$-groups $G$ with arbitrarily large nilpotence class. Other sets, such as $\{1\}$ and $\{1, p^e\}$ for $e > 1$ occur as $\text{c.d.}(G)$ only for groups with bounded class. (It is proved in Theorem 3.10 of [1] that if $\text{c.d.}(G) = \{1, p^e\}$ with $e > 1$, then $G$ has class at most $p$.) It would be interesting to be able to determine for an arbitrary finite set of $p$-powers whether it is of “bounded class type” or “unbounded class type”. We shall not, however, explore that question further here.

2. Our key lemma is the following.

Lemma. Let $U$ be an abelian group which acts on an abelian group $A$ and let $\mathcal{S}$ be the set of the sizes of the orbits in this action. Write $G = \hat{A} \rtimes A$, where $\hat{A}$ is the “dual group” (i.e. the group of linear characters) of $A$ and the semidirect product is constructed with respect to the action of $U$ on $\hat{A}$ induced by the given action of $U$ on $A$. Then $\text{c.d.}(G) = \mathcal{S}$. Furthermore, if $[A, U, U] = 1$, then $G$ has nilpotence class $\leq 2$.

Proof. Let $\lambda$ be any linear character of the group $\hat{A}$ and let $T$ be the stabilizer of $\lambda$ in $U$. Since $\hat{A}/(\ker \lambda)$ is centralized by $T$, which is abelian, it follows that $\hat{A}T/(\ker \lambda)$ is abelian and thus every $\psi \in \text{Irr}(\hat{A}T|\lambda)$ is linear. Now $\hat{A}T = I_G(\lambda)$ and

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thus character induction defines a bijection \( \text{Irr}(A^T|\lambda) \to \text{Irr}(G|\lambda) \) by Theorem 6.11 of \([2]\). It follows that every \( \chi \in \text{Irr}(G|\lambda) \) has degree \( |G : A^T| = |U : T| \). Therefore \( \text{c.d.}(G) = \{|U : T| \mid T \text{ is the stabilizer in } U \text{ of some } \lambda \in \text{Irr}(A)\} \). This establishes that \( \text{c.d.}(G) \) is equal to the set of orbit sizes in the action of \( U \) on \( \text{Irr}(A) \). However, there is a natural correspondence between \( \text{Irr}(A) \) and \( A \) and this defines a permutation isomorphism of the actions of \( U \) on \( A \) and \( \text{Irr}(A) \) and we conclude that \( \text{c.d.}(G) = \mathcal{P} \) as required.

Now suppose \( [A, U, U] = 1 \). If \( a \in A \), we have for \( u \in U \) and \( a \in A \) that
\[
[a, u](a) = (a^{-1}au)(a) = a(a^{-1})a(ua^{-1}) = \alpha([a, u^{-1}]).
\]
Therefore, if \( v \in U \), we get
\[
[a, u, v](a) = [a, u][(a, v^{-1})] = \alpha([a, v^{-1}, u^{-1}]) = 1
\]
and so \( [A, U, U] = 1 \). Since \( A \) is abelian, this yields \( [A, U] \subseteq Z(G) \). Also, \( G/[A, U] \) is abelian since \( G = AU \) and both factors are abelian. Therefore, \( G' \subseteq Z(G) \) and \( G \) is nilpotent with class \( \leq 2 \). \( \square \)

3. We now state and prove our main result.

**Theorem.** Let \( p \) be prime and let \( 0 = e_0 < e_1 < \cdots < e_m \) be integers. Then there exists a \( p \)-group \( G \) with nilpotence class \( \leq 2 \) such that
\[
\text{c.d.}(G) = \{p^i \mid 0 \leq i \leq m\}.
\]

**Proof.** Let \( U \) be an elementary abelian \( p \)-group of rank \( e_m \) with generators \( u_1, u_2, \ldots, u_{e_m} \). Let \( A \) be an elementary abelian \( p \)-group with basis \( \{a_i\} \cup \{z_{ij}\} \) for \( 1 \leq i \leq m \) and \( 1 \leq \mu < e_i \) and define an action of \( U \) on \( A \) as follows.

Put \( (z_{ij})^u = z_{ij} \) for all \( i, \mu, v \) and
\[
(a_i)^u = \begin{cases} a_i & \text{if } v > e_i, \\ a_i z_{ij} & \text{if } v \leq e_i. \end{cases}
\]
Since the automorphisms of \( A \) defined this way all have order \( p \) and commute pairwise, this does define an action of \( U \) on \( A \).

Our next task is to compute the sizes of the orbits of this action. Write \( Z = \langle z_{ij} \mid \mu \leq e_i \rangle \subseteq A \) and let \( a \in A \). If \( a \in Z \), it is in an orbit of size 1 and so we assume \( a \in A - Z \). There exists, then, a unique subscript \( i \) such that we can write \( a = bcz \), where \( b \in \langle a_j \mid j < i \rangle \), \( 1 \neq c \in \langle a_i \rangle \) and \( z \in Z \).

We claim that \( C_U(a) = \langle u_v \mid v > e_i \rangle \). Certainly, all \( u_v \) with \( v > e_i \) centralize all \( a_j \) with \( j < i \) and so they centralize \( a \). Conversely, suppose \( u \in U \). If \( u \) involves the generator \( u_\mu \), with \( \mu < e_i \), then since \( a \) involves \( a_\mu \), the exponents of \( z_{ij} \) in \( a \) and in \( a_\mu \) will not be equal and \( u \) does not centralize \( a \). This establishes the claim.

We now have \( |U: C_U(a)| = p^{e_i} \) and we see that the orbit sizes of the action of \( U \) on \( A \) are precisely the numbers \( p^{e_i} \) for \( 0 \leq i \leq m \). Since \( [A, U] \subseteq Z \), we have \( [A, U, U] = 1 \) and the result follows by the lemma. \( \square \)

**References**


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