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SETS OF p-POWERS AS IRREDUCIBLE CHARACTER DEGREES

I. M. ISAACS[1]

ABSTRACT. In this paper it is shown that every finite set of powers of the prime p which contains $p^0 = 1$ occurs as the full set of degrees of the irreducible characters of some p-group.

1. Let G be a finite group and write c.d.(G) to denote the set of numbers which occur as degrees of irreducible complex characters of G. There has been occasional interest in the question of which finite sets of positive integers can occur as c.d.(G). An obvious necessary condition is that 1 must lie in the set. This is not, however, sufficient since, for instance, the set $\{1, 2, 3, 5\}$ is not c.d.(G) for any G. (This follows from Theorem 6.1 of **[1]** where it is shown that if c.d.(G) consists of just 1 and prime numbers, then at most two primes occur.)

The purpose of this note is to construct examples which show that for each prime p, every finite set of powers of p which contains 1 occurs as c.d.(G) for some p-group G with nilpotence class $\leqslant 2$.

Some sets of powers of p, such as $\{1, p\}$, can occur as c.d.(G) for p-groups G with arbitrarily large nilpotence class. Other sets, such as $\{1\}$ and $\{1, p^e\}$ for $e > 1$ occur as c.d.(G) only for groups with bounded class. (It is proved in Theorem 3.10 of **[1]** that if c.d.$(G) = \{1, p^e\}$ with $e > 1$, then G has class at most p.) It would be interesting to be able to determine for an arbitrary finite set of p-powers whether it is of "bounded class type" or "unbounded class type". We shall not, however, explore that question further here.

2. Our key lemma is the following.

LEMMA. *Let U be an abelian group which acts on an abelian group A and let $\mathcal{S}$ be the set of the sizes of the orbits in this action. Write $G = \hat{A} \rtimes U$, where $\hat{A}$ is the "dual group" (i.e. the group of linear characters) of A and the semidirect product is constructed with respect to the action of U on $\hat{A}$ induced by the given action of U on A. Then* c.d.$(G) = \mathcal{S}$. *Furthermore, if $[A, U, U] = 1$, then G has nilpotence class $\leqslant 2$.*

PROOF. Let λ be any linear character of the group $\hat{A}$ and let T be the stabilizer of λ in U. Since $\hat{A}/(\ker \lambda)$ is centralized by T, which is abelian, it follows that $\hat{A}T/(\ker\ \lambda)$ is abelian and thus every $\psi \in \mathrm{Irr}(\hat{A}T|\lambda)$ is linear. Now $\hat{A}T = I_G(\lambda)$ and

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thus character induction defines a bijection $\operatorname{Irr}(\hat{A}T|\lambda) \to \operatorname{Irr}(G|\lambda)$ by Theorem 6.11 of [2]. It follows that every $\chi \in \operatorname{Irr}(G|\lambda)$ has degree $|G : \hat{A}T| = |U : T|$. Therefore $\text{c.d.}(G) = \{|U : T| \mid T$ is the stabilizer in U of some $\lambda \in \operatorname{Irr}(\hat{A})\}$. This establishes that c.d.(G) is equal to the set of orbit sizes in the action of U on $\operatorname{Irr}(\hat{A})$. However, there is a natural correspondence between $\operatorname{Irr}(\hat{A})$ and A and this defines a permutation isomorphism of the actions of U on A and $\operatorname{Irr}(\hat{A})$ and we conclude that c.d.$(G) = \mathscr{S}$ as required.

Now suppose $[A, U, U] = 1$. If $\alpha \in \hat{A}$, we have for $u \in U$ and $a \in A$ that

$$[\alpha, u](a) = (\alpha^{-1}\alpha^{u})(a) = \alpha(a^{-1})\alpha(uau^{-1}) = \alpha([\alpha, u^{-1}]).$$

Therefore, if $v \in U$, we get

$$[\alpha, u, v](a) = [\alpha, u]([a, v^{-1}]) = \alpha([a, v^{-1}, u^{-1}]) = 1$$

and so $[\hat{A}, U, U] = 1$. Since $\hat{A}$ is abelian, this yields $[\hat{A}, U] \subseteq \mathbf{Z}(G)$. Also, $G/[\hat{A}, U]$ is abelian since $G = \hat{A}U$ and both factors are abelian. Therefore, $G' \subseteq \mathbf{Z}(G)$ and G is nilpotent with class $\leqslant 2$. □

3. We now state and prove our main result.

THEOREM. *Let p be prime and let $0 = e_0 < e_1 < \cdots < e_m$ be integers. Then there exists a p-group G with nilpotence class $\leqslant 2$ such that*

$$\text{c.d.}(G) = \{p^{e_i} | 0 \leqslant i \leqslant m\}.$$

PROOF. Let U be an elementary abelian p-group of rank e_m with generators $u_1, u_2, \ldots, u_{e_m}$. Let A be an elementary abelian p-group with basis $\{a_i\} \cup \{z_{i\mu}\}$ for $1 \leqslant i \leqslant m$ and $1 \leqslant \mu \leqslant e_i$ and define an action of U on A as follows.

Put $(z_{i\mu})^{u_\nu} = z_{i\mu}$ for all i, μ, ν and

$$(a_i)^{u_\nu} = \begin{cases} a_i & \text{if } \nu > e_i, \\ a_i z_{i\nu} & \text{if } \nu \leqslant e_i. \end{cases}$$

Since the automorphisms of A defined this way all have order p and commute pairwise, this does define an action of U on A.

Our next task is to compute the sizes of the orbits of this action. Write $Z = \langle z_{i\mu} | \mu \leqslant e_i \rangle \subseteq A$ and let $a \in A$. If $a \in Z$, it is in an orbit of size 1 and so we assume $a \in A - Z$. There exists, then, a unique subscript i such that we can write $a = bcz$, where $b \in \langle a_j | j < i \rangle$, $1 \neq c \in \langle a_i \rangle$ and $z \in Z$.

We claim that $\mathbf{C}_U(a) = \langle u_\nu | \nu > e_i \rangle$. Certainly, all u_ν with $\nu > e_i$ centralize all a_j with $j \leqslant i$ and so they centralize a. Conversely, suppose $u \in U$. If u involves the generator u_μ, with $\mu \leqslant e_i$, then since a involves a_i, the exponents of $z_{i\mu}$ in a and in a^u will not be equal and u does not centralize a. This establishes the claim.

We now have $|U : \mathbf{C}_U(a)| = p^{e_i}$ and we see that the orbit sizes of the action of U on A are precisely the numbers p^{e_i} for $0 \leqslant i \leqslant m$. Since $[A, U] \subseteq Z$, we have $[A, U, U] = 1$ and the result follows by the lemma. □

REFERENCES

1. I. M. Isaacs and D. S. Passman, *A characterization of groups in terms of the degrees of their characters.* II, Pacific J. Math. **24** (1968), 467–510.

2. I. M. Isaacs, *Character theory of finite groups*, Academic Press, New York, 1976.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706