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PROPERTIES OF ENDOMORPHISM RINGS OF MODULES AND THEIR DUALS

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ABSTRACT. Let ${}_RM$ be a nonsingular left R-module whose Morita context is nondegenerate, let $B = \mathrm{End}_R M$ and let $M^* = \mathrm{Hom}_R(M, R)$. We show that B is left (right) strongly modular if and only if any element of B which has zero kernel in ${}_RM$ (M_R^*) has essential image in ${}_RM$ (M_R^*), and that B is a left (right) Utumi ring if and only if every submodule ${}_RU$ of ${}_RM$ (U_R^* of M_R^*) such that $U^\perp = 0$ (${}^\perp U^* = 0$) is essential in ${}_RM$ (M_R^*).

1. Introduction. Let ${}_RM$ be a left R-module whose standard Morita context is nondegenerate (see Definition 1); let $B = \mathrm{End}_R M$ be the ring of R-endomorphisms of ${}_RM$ and let $M^* = \mathrm{Hom}_R(M, R)$ be its dual module. Then B is left nonsingular if and only if ${}_RM$ is nonsingular (i.e. M satisfies the following: any $m \in M$ with essential annihilator in R must be zero), and B is right nonsingular if and only if M_R^* satisfies the following condition: If U_R^* is an essential submodule of M_R^* then the annihilator of U^* in B must be zero (Proposition 5). This condition certainly holds if M_R^* is nonsingular. Of course, just as for ${}_RM$, M_R^* is nonsingular if and only if $\mathrm{End}_R M^*$ is right nonsingular. Our concern, however, is with B, which is in general —for example for a nonfinitely generated ${}_RM$—a proper subring of $\mathrm{End}_R M^*$; hence a condition on ${}_RM$ which is equivalent to a certain left property of B is not expected to be equivalent to the same right property of B when it is reflected in M_R^*. In this paper, we investigate this situation and try to pick out some left-right properties of B which are symmetrically, or almost symmetrically, represented on ${}_RM$ and M_R^*. For example, we find that B is left strongly modular if and only if any element of B which has zero kernel in ${}_RM$ has essential image in ${}_RM$, while B is right strongly modular if and only if any element of B which has zero kernel in M_R^* has essential image in M_R^* (Theorem 3); and we find that B is a left Utumi ring if and only if every submodule ${}_RU$ of ${}_RM$ such that $U^\perp = 0$ is essential in ${}_RM$, while B is a right Utumi ring if and only if every submodule U_R^* of M_R^* such that ${}^\perp U^* = 0$ is essential in M_R^* (Theorem 7). These conditions naturally raise the general question of how B sits in $\mathrm{End}_R M^*$, a question which we do not treat in this paper, but which we expect to investigate in a future article.

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2. Preliminaries. The left and right annihilators in B of a subset K of B will be denoted by $\mathscr{L}(K)$ and $\mathscr{R}(K)$, respectively. The notation $l_M(K)$, $r_{M^*}(K)$, $r_B(U)$, $l_B(U^*)$ will be used for the annihilators in M of $K \subseteq B$ in M^* of $K \subseteq B$, in B of $U \subseteq M$ and in B of $U^* \subseteq M^*$, respectively. Also, for ${}_RU \subseteq {}_RM$ and $U_R^* \subseteq M_R^*$, we will use: $I_B(U) = \{b \in B\colon Mb \subseteq U\}$ and $I_B(U^*) = \{b \in B\colon bM^* \subseteq U^*\}$. The notation ${}_RU \subset {}^e{}_RM$ will be used to indicate that U is an essential R-submodule of M, i.e. U intersects nontrivially every nonzero R-submodule of M. Recall that ${}_RM$ is said to be *nonsingular* in case, for $m \in M$, $l_R(m) \subset {}^e{}_RR \Rightarrow m = 0$; B is said to be left (right) nonsingular if ${}_BB$ (B_B) is nonsingular.

We recall the following definition and proposition from [**4**]:

DEFINITION 1. Let (R, M, N, S) be a Morita context; that is, let ${}_RM_S$ and ${}_SN_R$ be bimodules with an R-R bimodule homomorphism $(\ ,\ )$: $M \otimes_S N \to R$ and an S-S bimodule homomorphism $[\ ,\ ]$: $N \otimes_R M \to S$ satisfying

$$m_1[n_1, m_2] = (m_1, n_1)m_2 \quad \text{and} \quad n_1(m_1, n_2) = [n_1, m_1]n_2$$

for all m_1, $m_2 \in M$ and n_1, $n_2 \in N$.

Then (R, M, N, S) is said to be *nondegenerate* if and only if the four modules ${}_RM$, M_S, ${}_SN$, N_R and the two pairings are faithful (the latter meaning that $(m, N) = 0$ implies $m = 0$, and three analogous implications).

This is equivalent to the eight natural maps associated with the Morita context being injective (for example, two of these maps are: $m \mapsto (m, -)$ and $r \mapsto (n \mapsto nr) \in \operatorname{End}({}_SN)$, for $m \in M$, $n \in N$ and $r \in R$). The standard context (R, M, M^*, B) of a module ${}_RM$ is nondegenerate if and only if ${}_RM$ is torsionless and faithful and the right annihilator of trace $({}_RM)$ is zero. We shall call such a module—i.e. one whose standard context is nondegenerate—a nondegenerate module, for brevity.

PROPOSITION 1 [**4**, PROPOSITION 14]. *If the context (R, M, N, S) is nondegenerate, and if one of ${}_RR$, ${}_RM$, ${}_SN$, ${}_SS$ is nonsingular, then all of them are nonsingular.*

Henceforth, unless otherwise indicated, let ${}_RM$ be a nondegenerate, nonsingular left R-module. Then, by the preceding, ${}_RM$, M_B, ${}_BM^*$, M_R^* and the two pairings are faithful, and ${}_RR$, ${}_RM^*$ and ${}_BB$ are nonsingular. $(\ ,\ )$ and $[\ ,\ ]$ will denote the pairings associated with the standard context for ${}_RM$, i.e. $(\ ,\ )$ is defined by $(m, f) = mf$ for $m \in M$ and $f \in M^*$, and $[f, m]$ is defined by $m_1[f, m] = (m_1, f)m$ for all m, $m_1 \in M$ and $f \in M^*$.

If ${}_RU$ is a submodule of ${}_RM$ then $[M^*, U]$ indicates the left ideal of B: $[M^*, U] = \{\sum_{i=1}^{s}[m_i^*, u_i]\colon m_i^* \in M^*, u_i \in U_i\}$, and similarly for $[U^*, M]$ where U_R^* is a submodule of M_R^*. Also, $U^{\perp} = \{m^* \in M^*\colon (U, m^*) = 0\}$ and ${}^{\perp}U^* = \{m \in M\colon (m, U^*) = 0\}$.

The well-known fact that, for a nonsingular module ${}_RM$, any R-homomorphism f, to ${}_RM$ from any other R-module, which has essential kernel is zero, will be used repeatedly without comment.

The following lemma will be useful to us in the sequel.

LEMMA 2. *For $K \subseteq B$, $\mathscr{L}(K) = I_Bl_M(K) = l_B(KM^*)$, and $\mathscr{R}(K) = r_B(MK) = I_B^*r_{M^*}(K)$.*

PROOF.

$b \in \mathscr{L}(K) \Leftrightarrow bK = 0 \Leftrightarrow MbK = 0 \Leftrightarrow Mb \subseteq l_M(K) \Leftrightarrow b \in I_B l_M(K)$;

$b \in \mathscr{L}(K) \Leftrightarrow bK = 0 \Leftrightarrow bKM^* = 0 \Leftrightarrow b \in l_B(KM^*)$;

$b \in \mathscr{R}(K) \Leftrightarrow Kb = 0 \Leftrightarrow MKb = 0 \Leftrightarrow b \in r_B(MK)$;

$b \in \mathscr{R}(K) \Leftrightarrow Kb = 0 \Leftrightarrow KbM^* = 0 \Leftrightarrow bM^* \subseteq r_{M^*}(K) \Leftrightarrow b \in I_B^* r_{M^*}(K)$. □

3. Strongly modular and Utumi endomorphism rings. In [2], a Baer *-ring B is called *strongly modular* in case, for all b in B, $\mathscr{R}(b) = 0$ implies that bB is essential in B. Because of the involution, the definition is left-right symmetric. In the absence of an involution, call a ring B *left strongly modular* if, for $b \in B$, $\mathscr{L}(b) = 0 \Rightarrow Bb \subset^e {}_B B$, and *right strongly modular* if $\mathscr{R}(b) = 0 \Rightarrow bB \subset^e B_B$. It turns out that the properties of left and right strong modularity of $B = \operatorname{End}_R M$ are equivalent to almost symmetric conditions on ${}_R M$ and M_R^*.

THEOREM 3. (i) *B is left strongly modular if and only if, for each $b \in B$, $l_M(b) = 0 \Rightarrow Mb \subset^e {}_R M$;*

(ii) *B is right strongly modular if and only if, for each $b \in B$, $r_{M^*}(b) = 0 \Rightarrow bM^* \subset^e M_R^*$.*

PROOF. By comparing the definition of left strong modularity with the condition on ${}_R M$ in (i), it is easily seen that (i) will follow as soon as we show that "$\mathscr{L}(b) = 0$" is equivalent to "$l_M(b) = 0$" and that "$Bb \subset^e {}_B B$" is equivalent to "$Mb \subset^e {}_R M$"; these equivalences will be proved in Lemma 4 which follows. Similarly, (ii) will follow once we show, in Lemma 4, that "$\mathscr{R}(b) = 0$" is equivalent to "$r_{M^*}(b) = 0$" and that "$bB \subset^e B_B$" is equivalent to "$bM^* \subset^e M_R^*$".

LEMMA 4. (i) *For any subset K of B, $\mathscr{L}(K) = 0$ if and only if $l_M(K) = 0$ and $\mathscr{R}(K) = 0$ if and only if $r_{M^*}(K) = 0$.*

(ii) *For any left ideal ${}_B H$ of B, ${}_B H \subset^e {}_B B$ if and only if $MH \subset^e {}_R M$; and for any right ideal J_B of B, $J_B \subset^e B_B$ if and only if $JM^* \subset^e M_R^*$.*

PROOF. (i) Let $K \subseteq B$ and consider the submodule $l_M(K)$ of ${}_R M$. If $l_M(K) \neq 0$, let $0 \neq m \in l_M(K)$; then, by nondegeneracy, there is $m^* \in M^*$ such that $[m^*, m] \neq 0$. Then, since M_B is faithful, $0 \neq M[m^*, m] = (M, m^*)m \subseteq Rm \subseteq l_M(K)$; that is, $0 \neq [m^*, m] \in I_B l_M(K)$. Hence, $l_M(K) = 0$ if and only if $\mathscr{L}(K) = I_B l_M(K) = 0$.

Similarly, if $0 \neq m^* \in r_{M^*}(K)$, then nondegeneracy gives $m \in M$ such that $[m^*, m] \neq 0$, and since ${}_B M^*$ is faithful, $[m^*, m]$ is a nonzero element of $I_B^* r_{M^*}(K)$; hence $r_{M^*}(K) = 0$ if and only if $\mathscr{R}(K) = I_B^* r_{M^*}(K) = 0$.

(ii) Let ${}_B H$ be an essential left ideal of B and let $0 \neq m \in M$. Then $[M^*, m] \cap H \neq 0$, and, since M_B is faithful,

$$0 \neq M([M^*, m] \cap H) \subseteq M[M^*, m] \cap MH = (M, M^*)m \cap MH \subseteq Rm \cap MH,$$

proving that $MH \subset^e {}_R M$.

Conversely, assume that $MH \subset^e {}_R M$ for some left ideal ${}_B H$ of B and let $0 \neq c \in B$. Then, since M_B is faithful, $Mc \neq 0$, and hence $Mc \cap MH \neq 0$. By nondegeneracy,

$$0 \neq [M^*, Mc \cap MH] \subseteq [M^*, Mc] \cap [M^*, MH] \subseteq Bc \cap [M^*, MH].$$

This shows that $[M^*, MH] \subset^e {}_B B$, and hence, since $[M^*, MH] \subseteq {}_B H$, we have ${}_B H \subset^e {}_B B$.

Similarly, if J is an essential right ideal of B and $0 \neq m^* \in M^*$, then, by nondegeneracy, $[m^*, M] \neq 0$, hence $[m^*, M] \cap J \neq 0$. Since M_B^* is faithful, this implies

$$
\begin{aligned}
0 \neq ([m^*, M] \cap J)M^* &\subseteq [m^*, M]M^* \cap JM^* \\
&= m^*(M, M^*) \cap JM^* \subseteq m^*R \cap JM^*;
\end{aligned}
$$

so $JM^* \subset^e M_R^*$.

Conversely, if $JM^* \subset^e M_R^*$ for some right ideal J_B of B, and $0 \neq c \in B$, then $JM^* \cap cM^* \neq 0$ and $[JM^* \cap cM^*, M] \neq 0$ by nondegeneracy; hence,

$$0 \neq [JM^* \cap cM^*, M] \subseteq [JM^*, M] \cap [cM^*, M] \subseteq [JM^*, M] \cap cB.$$

This implies that $[JM^*, M] \subset^e B_B$, and hence, since $[JM^*, M] \subseteq J$, we have $J_B \subset^e B_B$. □

REMARKS. 1. One property of nondegenerate modules that can be deduced from the proof of Lemma 4 is that $I_B(U) = 0$ if and only if $U = 0$ for a submodule ${}_R U$ of ${}_R M$, and similarly for $U_R^* \subseteq M_R^*$.

2. In the proof of Lemma 4(ii), we have shown that ${}_R U \subset^e {}_R M \Rightarrow [M^*, U] \subset^e {}_B B$ and $U_R^* \subset^e M_R^* \Rightarrow [U^*, M] \subset^e B_B$.

Aside from completing the proof of Theorem 3, Lemma 4 is also useful in giving a condition on M_R^* which is equivalent to right nonsingularity of B, as in the next result.

PROPOSITION 5. *B is right nonsingular if and only if, for any submodule U_R^* of M_R^*, $U_R^* \subset^e M_R^* \Rightarrow l_B(U^*) = 0$.*

PROOF. It was shown in [3, Proposition 1] that, under our present hypotheses, B is right nonsingular if and only if, for any submodule ${}_R U$ of ${}_R M$, $r_B(U) \subset^e B_B \Rightarrow U = 0$.

Assume that B is right nonsingular and suppose that $U_R^* \subset^e M_R^*$; then, as noted in Remark 2 above, $[U^*, M] \subset^e B_B$. We have $(Ml_B(U^*))[U^*, M] = (Ml_B(U^*), U^*)M = 0$; therefore $[U^*, M] \subseteq r_B(Ml_B(U^*))$, which implies $r_B(Ml_B(U^*)) \subset^e B_B$. Hence, by [3, Proposition 1], since B is right nonsingular, this implies that $Ml_B(U^*) = 0$; hence, since M_B is faithful, we have $l_B(U^*) = 0$.

Conversely, assume that $U_R^* \subset^e M_R^*$ implies $l_B(U^*) = 0$. Suppose that ${}_R U$ is a submodule of ${}_R M$ such that $r_B(U) \subset^e B_B$. Then, by Lemma 4(ii), $U_R^* = r_B(U)M^* \subset^e M_R^*$. Hence, by hypothesis, $l_B(r_B(U)M^*) = 0$. But $I_B(U) \subseteq l_B(r_B(U)M^*)$ since, always, $I_B(U)r_B(U) = 0$; hence $I_B(U) = 0$, which, by nondegeneracy (see Remark 1), implies that $U = 0$, completing the proof. □

A ring B is said to be a *left Utumi ring* in case, for any left ideal ${}_B H$ of B, $\mathscr{R}({}_B H) = 0 \Rightarrow {}_B H \subset^e {}_B B$; B is called a *right Utumi ring* if, for any right ideal J_B of B, $\mathscr{L}(J_B) = 0 \Rightarrow J_B \subset^e B_B$. In [2], it is shown that a strongly modular Baer *-ring is left and right Utumi [2, Theorem 2.3]. In our situation, i.e. for $B = \operatorname{End}_R M$, where ${}_R M$ is nondegenerate and nonsingular, it is easily shown that a left and right strongly modular Baer ring satisfies the Utumi conditions for principal left and right ideals. In fact, B need not be a Baer ring to show this; rather, left and right nonsingularity of B is sufficient, with left and right strong modularity, in order to obtain the Utumi conditions for principal ideals, as can be seen in Proposition 6. For the full Utumi conditions on B, however, we can give, in Theorem 7, conditions on ${}_RM$ and M_R^* which appear quite symmetrical.

PROPOSITION 6. *If ${}_RM$ is such that $B = \mathrm{End}_RM$ is a left and right nonsingular, left and right strongly modular ring, then*
(i) $\mathscr{L}(bB) = 0 \Rightarrow bB \subset^e B_B$, *and* (ii) $\mathscr{R}(Bb) = 0 \Rightarrow Bb \subset^e {}_BB$.

PROOF. (i)

$$\mathscr{L}(bB) = \mathscr{L}(b) = 0 \Rightarrow Bb \subset^e {}_BB \quad \text{since } B \text{ is left strongly modular,}$$
$$\Rightarrow \mathscr{R}(Bb) = 0 \quad \text{since } B \text{ is left nonsingular,}$$
$$\Rightarrow bB \subset^e B_B \quad \text{since } B \text{ is right strongly modular.}$$

(ii)

$$\mathscr{R}(Bb) = \mathscr{R}(b) = 0 \Rightarrow bB \subset^e B_B \quad \text{since } B \text{ is right strongly modular,}$$
$$\Rightarrow \mathscr{L}(bB) = 0 \quad \text{since } B \text{ is right nonsingular,}$$
$$\Rightarrow Bb \subset^e {}_BB \quad \text{since } B \text{ is left strongly modular.} \quad \square$$

For our last result, ${}_RM$ is assumed to satisfy the standing hypothesis, i.e. ${}_RM$ is nonsingular and nondegenerate.

THEOREM 7. (i) $B = \mathrm{End}_RM$ *is a left Utumi ring if and only if, for any submodule* ${}_RU$ *of* ${}_RM$, $U^\perp = 0 \Rightarrow {}_RU \subset^e {}_RM$; *and* (ii) $B = \mathrm{End}_RM$ *is a right Utumi ring if and only if, for any submodule* U_R^* *of* M_R^*, ${}^\perp U^* = 0 \Rightarrow U_R^* \subset^e M_R^*$.

PROOF. (i) Assume that B is a left Utumi ring; then, by [3, Lemma 3], we have, for any submodule ${}_RX$ of ${}_RM$, $r_B({}_RX) = 0 \Rightarrow {}_RX \subset^e {}_RM$. Let ${}_RU$ be a submodule of ${}_RM$ such that $U^\perp = 0$. Then $b \in r_B(U) \Rightarrow Ub = 0 \Rightarrow (U, bm^*) = (Ub, m^*) = 0$, for each $m^* \in M^*$, $\Rightarrow bm^* = 0$ since $U^\perp = 0$; but this means $bM^* = 0$, therefore, since ${}_BM^*$ is faithful, $b = 0$. Hence $r_B(U) = 0$, which implies, since B is left Utumi, that ${}_RU \subset^e {}_RM$.

Conversely, assume that ${}_RU^\perp = 0 \Rightarrow {}_RU \subset^e {}_RM$ for every ${}_RU \subseteq {}_RM$. Let ${}_BH$ be a left ideal of B with $\mathscr{R}({}_BH) = 0$. If $(MH, m^*) = 0$, then $(M, Hm^*) = 0$, which implies $Hm^* = 0$ by nondegeneracy. Then $[Hm^*, M] = 0$, i.e. $H[m^*, M] = 0$, which implies $[m^*, M] = 0$ since $\mathscr{R}(H) = 0$. Again by nondegeneracy, $[m^*, M] = 0 \Rightarrow m^* = 0$. Hence, we have shown that $(MH, m^*) = 0 \Rightarrow m^* = 0$, i.e. $(MH)^\perp = 0$, which by hypothesis implies that $MH \subset^e {}_RM$. Now, by Lemma 4(ii), this gives ${}_BH \subset^e {}_BB$, and B is left Utumi.

(ii) Assume that B is a right Utumi ring. Let U_R^* be a submodule of M_R^* such that ${}^\perp U^* = 0$. Consider the right ideal $[U^*, M]$ of B. If $m[U^*, M] = 0$, then $(m, U^*)M = 0$, hence, since ${}_RM$ is faithful, $(m, U^*) = 0$, which gives $m = 0$ since ${}^\perp U^* = 0$. Therefore, $l_M([U^*, M]) = 0$, hence, by Lemma 4(i), $\mathscr{L}([U^*, M]) = 0$, which implies that $[U^*, M] \subset^e B_B$ since B is right Utumi. Then, by Lemma 4(ii), $[U^*, M]M^* \subset^e M_R^*$. But $[U^*, M]M^* \subseteq U^*$, hence $U_R^* \subset^e M_R^*$, and we have shown that ${}^\perp U^* = 0$ implies $U_R^* \subset^e M_R^*$.

Conversely, assume that ${}^{\perp}U^* = 0 \Rightarrow U_R^* \subset^e M_R^*$ for any submodule U_R^* of M_R^*. Let J_B be a right ideal of B such that $\mathscr{L}(J_B) = 0$. Then, by Lemma 4(i), $l_M(J) = 0$; hence, if $(m, JM^*) = 0$, then $mJ = 0$ by nondegeneracy, and $m = 0$ since $l_M(J) = 0$. Thus, ${}^{\perp}(JM^*) = 0$, which, by hypothesis, implies that $JM^* \subset^e M_R^*$. Finally, by Lemma 4(ii), $JM^* \subset^e M_R^* \Rightarrow J_B \subset^e B_B$, completing the proof that B is right Utumi. □

Remarks. 1. The nondegeneracy condition on ${}_RM$ cannot be deleted from the hypothesis of Theorem 7, as we shall see in the following example.

First recall that a CS module is one in which every complement (= essentially closed) submodule is a direct summand, with a ring R being left or right CS whenever ${}_RR$ or R_R is a CS module. In [**1**], an example is given of a nonsingular, projective CS module P whose endomorphism ring, $B = \operatorname{End} P$, is not left CS (Example 3.3 in [**1**]). We will show that, for such a P, the condition "$U^{\perp} = 0 \Rightarrow U \subset^e P$, for any submodule U of P" of Theorem 7(i) does hold, and yet $B = \operatorname{End} P$ is not left Utumi, the reason being that the nondegeneracy condition does not hold in P.

Assume that $U^{\perp} = 0$ for a submodule U of P. Then, $b \in r_B(U) \Rightarrow Ub = 0 \Rightarrow (U, bP^*) = (Ub, P^*) = 0 \Rightarrow bP^* = 0$ since $U^{\perp} = 0$, and this last gives $b = 0$ since ${}_BP^*$ is faithful, which shows that $r_B(U) = 0$. Now, since P is a CS module, the essential-closure, U^e, of U is a direct summand in P, say $P = U^e \oplus V$, and there is an idempotent $b \in B$ such that $U^eb = 0$ and $vb = v$ for $v \in V$; then $r_B(U) = 0$ implies that $b = 0$, so $V = 0$ and $U \subset^e P$.

To see that B is not left Utumi, recall first that a ring is left nonsingular, left CS if and only if it is Baer and left Utumi (cf. e.g. [**1**, Theorem 2.1]); thus, since B is not left CS, it will suffice to show that B is Baer: Let J be any subset of B, then the essential closure, $(PJ)^e$, of PJ is a direct summand in P since P is CS, say $P = (PJ)^e \oplus U$; then, letting e be the idempotent in B with $\ker e = (PJ)^e$, we have $\mathscr{R}(J) = r_B(PJ) = r_B((PJ)^e) = eB$, which proves that B is a Baer ring.

Finally, to see that nondegeneracy of P does not hold, we remark that (a) P nondegenerate $\Rightarrow I_B(U) \neq 0$ for every nonzero submodule U of P, as noted in Remark 1 following Theorem 3; and (b) "$I_B(U) \neq 0$ for every $0 \neq U \subseteq P$" does not hold in P, because by Lemma 3 of [**3**] a nonsingular module with this property has a left Utumi endomorphism ring if and only if "$r_B(U) = 0 \Rightarrow U \subset^e P$", and we have just shown this last to be true in P, whereas B is not left Utumi.

2. In the special case when the nondegenerate, nonsingular ${}_RM$ is ${}_RR$, it is easy to see that the conditions in Theorem 7 are precisely the Utumi conditions for a left and right nonsingular R. We verify this for the left Utumi condition, by noting that "$U^{\perp} = 0$" becomes just "$r_B(U) = 0$" or "$\mathscr{R}(I) = 0$" for I a left ideal in B. For, in this case, $B = \operatorname{End}({}_RR) \cong R$; thus, if ${}_RU = {}_RI$ is a left ideal in R, then $I^{\perp} = 0 \Rightarrow r_B(I) = 0$: $b \in r_B(I) \Rightarrow Ib = 0 \Rightarrow (I, bR^*) = (Ib, R^*) = 0 \Rightarrow bR^* = 0$ since $I^{\perp} = 0$, $\Rightarrow b = 0$ since ${}_BR^*$ is faithful; and, conversely, $r_B(I) = 0 \Rightarrow I^{\perp} = 0$: $r^* \in I^{\perp} \Rightarrow (I, r^*) = 0 \Rightarrow Ir^*r = (I, r^*r) = 0$ for each $r \in R$, and this last implies that $r^*r = 0$ for each $r \in R$, when we consider r^*r as being in $R \cong B$ and use the fact that $r_B(I) = 0$; finally, $r^*R = 0 \Rightarrow r^* = 0$ since R_R^* is faithful.

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