

COUNTEREXAMPLE TO THE SPECTRAL MAPPING THEOREM FOR THE EXPONENTIAL FUNCTION¹

J. HEJTMANEK AND HANS G. KAPER

ABSTRACT. An example is given of an unbounded operator in a Hilbert space which generates a strongly continuous semigroup and for which the spectral mapping theorem for the exponential function does not hold. The spectra of both the generator and the semigroup are determined explicitly.

1. Introduction. Let X be a Banach space and $W = [W(t): t \geq 0]$ a strongly continuous semigroup of bounded linear operators on X with generator A , $W(t) = \exp(At)$. The spectral bound of the generator, s , is

$$s = \sup\{\operatorname{Re} \lambda: \lambda \in \sigma(A)\};$$

the type of the semigroup, ω_0 , is

$$\omega_0 = \inf\{\omega \in \mathbf{R}: \exists M \geq 1, \forall t \geq 0, \|W(t)\| \leq M \exp(\omega t)\}.$$

It is well known that $s \leq \omega_0$. Furthermore, $r(W(t)) = \exp(\omega_0 t)$, where r is the spectral radius.

In general, $\exp(\sigma(A)t) \subset \sigma(W(t)) \setminus \{0\}$. The identity

$$\exp(\sigma(A)t) = \sigma(W(t)) \setminus \{0\} \quad \text{for all } t \geq 0,$$

is known as the Spectral Mapping Theorem (SMT) for the exponential function. SMT is true if $[W(t): t \geq t_0]$ is uniformly continuous for some $t_0 > 0$ (see Davies [1980, Theorem 2.19]). SMT cannot be true if $s < \omega_0$.

Semigroups for which $s < \omega_0$ have been given by Hille and Phillips [1957, XXIII, 16], Foias [1973], Zabczyk [1975], Greiner, Voigt and Wolff [1981], Wolff [1981], Nagel [1982], and Hoppen [1985]. In each case, the spectrum of the generator could be computed, but not the spectrum of the semigroup; the type of the semigroup was computed from the definition. In this article we present an example of a semigroup with $s < \omega_0$, for which the spectra of both the generator and the semigroup can be computed. The example incorporates aspects of the examples of Foias and Zabczyk.

This research was motivated by our work in linear transport theory, where SMT plays an important role in the determination of the asymptotic behavior of particle distribution functions; see Kaper, Lekkerkerker and Hejtmanek [1982, Chapters 12 and 13], Greiner [1982], Voigt [1984], and Kaper and Hejtmanek [1984]. General

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references for the theory of linear semigroups are Hille and Phillips [1957], Davies [1980], Goldstein [1983], and Pazy [1983]; see also Dunford and Schwartz [1958, §VIII.1], Kato [1966, Chapter X], and Reed and Simon [1975, §X.8]. We shall make use of an SMT-related result of Gearhart [1978]; see also Herbst [1983], Howland [1984], and Pruess [1984].

2. Definitions and basic properties. Let $\{H_k: k = 2, 3, \dots\}$ be a sequence of Hilbert spaces and $H = \bigoplus_{k=2}^{\infty} H_k$. Let $B_k \in L(H_k)$ be such that $\|B_k\| = 1$ and $\sigma(B_k) = \{0\}$. We define $A_k \in L(H_k)$ by the expression

$$(2.1) \quad A_k = 2\pi i k I_k + B_k,$$

where I_k is the identity in $L(H_k)$, and we define the unbounded operator A on $\text{dom } A \subset H$ by the direct sum

$$(2.2) \quad A = \bigoplus_{k=2}^{\infty} A_k.$$

The operator A is closed and $\text{dom } A = \{x \in H: \sum_{k=2}^{\infty} k^2 \|x_k\|^2 < \infty\}$. The resolvent of A , which is given by

$$(2.3) \quad R(\lambda, A) = \bigoplus_{k=2}^{\infty} R(\lambda, A_k), \quad \lambda \in \rho(A),$$

satisfies the norm identity

$$(2.4) \quad \|R(\lambda, A)\| = \sup\{\|R(\lambda, A_k)\|: k = 2, 3, \dots\}.$$

The spectrum of A is easily computed. Because $2\pi i k \in \sigma(A_k)$ for each $k = 2, 3, \dots$, we have $2\pi i k \in \sigma(A)$ for $k = 2, 3, \dots$. For any point $\lambda \neq 2\pi i k$, $k = 2, 3, \dots$, we have

$$(2.5) \quad R(\lambda, A_k) = (\lambda - 2\pi i k)^{-1} (I_k - (\lambda - 2\pi i k)^{-1} B_k)^{-1}.$$

Because $\|B_k\| = 1$ for all values of k , and $|\lambda - 2\pi i k| \geq 1 + \varepsilon$, $\varepsilon > 0$, for all but at most one value k_0 of k , where $\lambda - 2\pi i k_0 \in \rho(B_{k_0})$, it follows that $\sup\{\|R(\lambda, A_k)\|: k = 2, 3, \dots\} < \infty$ and, therefore, $\lambda \in \rho(A)$. Hence,

$$(2.6) \quad \sigma(A) = \{2\pi i k: k = 2, 3, \dots\},$$

and $s = 0$.

If $\lambda \in \mathbf{C}$ and $|\lambda - 2\pi i k| > 1$, then $\|R(\lambda, A_k)\| < (|\lambda - 2\pi i k| - 1)^{-1}$. If, in addition, $\text{Re } \lambda > 1$, then $\|R(\lambda, A_k)\| \leq (\text{Re } \lambda - 1)^{-1}$ for $k = 2, 3, \dots$. Therefore,

$$(2.7) \quad \|R(\lambda, A)\| \leq (\text{Re } \lambda - 1)^{-1}, \quad \lambda \in \mathbf{C}, \text{Re } \lambda > 1.$$

It follows from the Hille-Yosida Theorem that A is the generator of a semigroup $W = [W(t): t \geq 0]$, with $W(t) = \exp(At)$ given by

$$(2.8) \quad W(t) = \bigoplus_{k=2}^{\infty} W_k(t), \quad t \geq 0,$$

where

$$(2.9) \quad W_k(t) = \exp(2\pi i k t) \exp(B_k t).$$

Clearly, $\|W(t)\| \leq e^t$ for $t \geq 0$, and $\omega_0 \leq 1$.

For special choices of H_k and B_k , Zabczyk showed that $\|W(t)\| = e^t$ for $t \geq 0$. Foias, choosing different H_k and B_k , showed that $e^{\alpha t} \in \sigma(W(t))$ for all $\alpha \in (0, 1)$. (Here we have changed the definitions to get agreement with Zabczyk's.) In both cases, therefore, $\omega_0 = 1$.

We take the same choices of H_k and B_k as Zabczyk:

$$(2.10) \quad H_k = \mathbf{C}^k, \quad B_k = \text{mat}[b_{k;i,j}; i, j = 1, \dots, k],$$

where $b_{k;i,i+1} = 1$ for $i = 1, \dots, k - 1$, $b_{k;i,j} = 0$ otherwise. One verifies that $\|B_k\| = 1$, $\sigma(B_k) = \{0\}$, and $B_k^k = 0$. The resolvent of B_k is given by a finite sum

$$(2.11) \quad R(\lambda, B_k) = \sum_{n=0}^{k-1} \lambda^{-(n+1)} B_k^n, \quad \lambda \in \mathbf{C}, \lambda \neq 0.$$

3. Bounds for the resolvent. Our aim is to apply the following theorem of Gearhart.

THEOREM 1. *Let $W = [W(t): t \geq 0]$ be a semigroup with generator A in a Hilbert space. Then $e^z \in \rho(W(1))$ iff*

- (i) $z + 2\pi il \in \rho(A)$ for all $l \in \mathbf{Z}$, and
- (ii) $\sup\{\|R(z + 2\pi il, A)\|: l \in \mathbf{Z}\} < \infty$.

PROOF. See Gearhart [1978], Herbst [1983], Howland [1984], and Pruess [1984].

Condition (i) is satisfied for any point $z \in \mathbf{C}$ that is not an integer multiple of $2\pi i$. Condition (ii) requires estimates of the resolvent along lines parallel to the imaginary axis. It follows from (2.4) and the definition of A_k that

$$(3.1) \quad \|R(z + 2\pi il, A)\| = \sup\{\|R(z + 2\pi i(l - k), B_k)\|: k = 2, 3, \dots\},$$

whenever $z + 2\pi il \in \rho(A)$.

LEMMA 2. *For all $\lambda \in \mathbf{C}$ with $|\lambda| > 0$ and for all $k = 2, 3, \dots$, we have the inequalities*

$$(3.2) \quad g(|\lambda|, k) \leq \|R(\lambda, B_k)\| \leq h(|\lambda|, k),$$

where

$$g(x, k) = \left(\sum_{j=1}^k x^{-2j}\right)^{1/2}, \quad h(x, k) = \sum_{j=1}^k x^{-j}$$

for $x > 0$ and $k = 2, 3, \dots$

PROOF. The upper bound in (3.2) is trivial. Let $e_k = (0, \dots, 0, 1)^T$. The vectors $B_k^n e_k$, $n = 0, \dots, k - 1$, are orthonormal in H_k , so

$$\|R(\lambda, B_k)\| \geq \|R(\lambda, B_k)e_k\| \geq \left(\sum_{j=1}^k |\lambda|^{-2j}\right)^{1/2}. \quad \square$$

The functions g and h have the following properties: (i) for x fixed, g and h increase as functions of k ; (ii) for k fixed, g and h decrease as functions of x ; (iii) $g(1, k) = \sqrt{k}$, $h(1, k) = k$; and (iv) for $x > 1$, $\lim_{k \rightarrow \infty} g(x, k) = (x^2 - 1)^{-1/2}$, $\lim_{k \rightarrow \infty} h(x, k) = (x - 1)^{-1}$. These properties imply that, for fixed k ,

$$(3.3) \quad \|R(\lambda, B_k)\| = |\lambda|^{-1} (1 + O(|\lambda|^{-1})) \quad \text{as } |\lambda| \rightarrow \infty,$$

$$(3.4) \quad \|R(\lambda, B_k)\| = |\lambda|^{-k} (1 + O(|\lambda|)) \quad \text{as } |\lambda| \rightarrow 0.$$

Furthermore, for $k = 2, 3, \dots$,

$$(3.5) \quad \|R(\lambda, B_k)\| \geq \sqrt{k} \quad \text{for all } |\lambda| \leq 1,$$

$$(3.6) \quad \|R(\lambda, B_k)\| \leq (\rho - 1)^{-1} \quad \text{for all } |\lambda| \geq \rho > 1.$$

In the next section we apply these estimates to calculate the supremum in (3.1).

4. Spectrum of the semigroup. Let $S = \bigcup_{m \in \mathbf{Z}} (D + 2\pi im)$, where D is the punctured unit disc, $D = \{z \in \mathbf{C}: 0 < |z| \leq 1\}$. Clearly, $S \subset \rho(A)$. If $z \in S$, then there exists, for every $k \in \{2, 3, \dots\}$, an $l \in \mathbf{Z}$ such that $|z + 2\pi i(l - k)| \leq 1$. Therefore, using (3.5), we see that

$$\sup\{\|R(z + 2\pi i(l - k), B_k)\|: l \in \mathbf{Z}, k = 2, 3, \dots\} = \infty,$$

and Theorem 1 yields $e^z \in \sigma(W(1))$.

If $z \in \mathbf{C} \setminus \bar{S}$, then $|z + 2\pi i(l - k)| \geq \text{dist}(z, \{2\pi im: m \in \mathbf{Z}\}) = \rho > 1$ for all $k = 2, 3, \dots$ and $l \in \mathbf{Z}$. Therefore, using (3.6), we see that

$$\sup\{\|R(z + 2\pi i(l - k), B_k)\|: l \in \mathbf{Z}, k = 2, 3, \dots\} < \infty,$$

and Theorem 1 yields $e^z \in \rho(W(1))$.

Combining these results, we find

$$(4.1) \quad \sigma(W(1)) \setminus \{0\} = e^{\sigma(A)} \cup \{e^z: 0 < |z| \leq 1\}.$$

Here, $e^{\sigma(A)}$ consists of the single point $\{e^z: z = 0\}$.

Let $t > 0$. Then Theorem 1 implies that $e^{tz} \in \rho(W(t))$ iff (i) $z + (2\pi/t)il \in \rho(A)$ for all $l \in \mathbf{Z}$, and (ii) $\sup\{\|R(z + (2\pi/t)il, A)\|: l \in \mathbf{Z}\} < \infty$. Here,

$$(4.2) \quad \|R(z + (2\pi/t)il, A)\| = \sup\{\|R(z + 2\pi i(l/t - k), B_k)\|: k = 2, 3, \dots\}.$$

We distinguish two cases: t rational and t irrational.

Let t be rational, $t = p/q$. Then

$$(4.3) \quad \sigma(W(p/q)) = (\sigma(W(1/q)))^p.$$

If $z \in S$ and $|z| \leq 1$, then there exists, for every $k \in \{q, 2q, \dots\}$, an $l \in \mathbf{Z}$ such that $|z + 2\pi i(ql - k)| \leq 1$. Therefore,

$$\sup\{\|R(z + 2\pi i(ql - k), B_k)\|: l \in \mathbf{Z}, k = 2, 3, \dots\} = \infty,$$

and $e^{tz} \in \sigma(W(t))$. A similar procedure yields $e^{tz} \in \sigma(W(t))$ if $z \in S$ and $|z + 2\pi im| \leq 1$ for some $m \in \mathbf{Z}$.

If $z \in \mathbf{C} \setminus \bar{S}$, then $|z + 2\pi i(ql - k)| \geq \text{dist}(z, \{2\pi im: m \in \mathbf{Z}\}) = \rho > 1$ for all $k = 2, 3, \dots$ and $l \in \mathbf{Z}$. Therefore,

$$\sup \{ \|R(z + 2\pi i(ql - k), B_k)\| : l \in \mathbf{Z}, k = 2, 3, \dots \} < \infty,$$

and $e^{tz} \in \rho(W(t))$.

Summarizing, we have

$$(4.4) \quad \sigma(W(t)) \setminus \{0\} = e^{\sigma(A)t} \cup \{e^{zt}: 0 < |z| \leq 1\},$$

whenever t is rational. If $t = p/q$, where p and q are relatively prime, then $e^{\sigma(A)t}$ is the finite set $\{e^{zt}: z = 2\pi ik, k = 0, \dots, q - 1\}$ of q points on the unit circle.

Let t be irrational. The sequence $\{l/t: l = 1, 2, 3, \dots\}$ is uniformly distributed modulo 1; cf. Cassels [1957, IV]. Hence, for every $z = \alpha + 2\pi i\beta$ with $0 < |\alpha| < 1$ and $0 \leq \beta \leq 1$, there exist infinitely many $k \in \{2, 3, \dots\}$ and $l \in \mathbf{Z}$ such that

$$|l/t - k + \beta| \leq (1 - \alpha^2)^{1/2}.$$

Each such pair (k, l) generates a point $z + 2\pi i(l/t - k)$ that belongs to D and therefore to S , so $e^{tz} \in \sigma(W(t))$. Furthermore, $e^{\sigma(A)t}$ is a dense set on the unit circle. Because $\sigma(W(t))$ is closed, it follows that

$$(4.5) \quad \sigma(W(t)) \setminus \{0\} = \{e^{tz}: |\text{Re } z| \leq 1, 0 \leq \text{Im } z \leq 2\pi\},$$

whenever t is irrational.

Thus, $\sigma(W(t))$ is known for all $t > 0$.

5. Remarks. Because $-A$ also generates a semigroup, A is the generator of a group, and

$$(5.1) \quad \|R(\lambda, A)\| \leq (|\text{Re } \lambda| - 1)^{-1}, \quad \lambda \in \mathbf{C}, |\text{Re } \lambda| > 1.$$

Furthermore,

$$(5.2) \quad \|R(\lambda, A)\| \leq |\text{Im } \lambda|^{-1}, \quad \lambda \in \mathbf{C}, \text{Im } \lambda < 0.$$

Hence, iA also generates a semigroup.

The graph of $\|R(\lambda, A)\|$, $\lambda \in \rho(A)$, has the following qualitative behavior. As $|\text{Re } \lambda| \rightarrow \infty$, it decreases to zero like $|\text{Re } \lambda|^{-1}$; as $\text{Im } \lambda \rightarrow -\infty$, it decreases to zero like $|\text{Im } \lambda|^{-1}$. It is bounded in the strip $\{\lambda \in \mathbf{C}: |\text{Re } \lambda| \leq 1, \text{Im } \lambda \geq 0\}$, except in the discs $D + 2\pi ik$, $k = 2, 3, \dots$, where $\|R(z, A)\| = |z - 2\pi ik|^{-k}(1 + O(|z - 2\pi ik|))$ as $z \rightarrow 2\pi ik$.

If A is rotated over an angle ϕ , $0 < \phi < \pi/2$, then we obtain a new operator $e^{i\phi}A$ for which SMT holds. This result is verified most easily from Theorem 1 if one uses the above qualitative results about the graph of $\|R(\lambda, A)\|$.

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MATHEMATICS AND COMPUTER SCIENCE DIVISION, ARGONNE NATIONAL LABORATORY, ARGONNE, ILLINOIS 60439 (Current address of H. G. Kaper)

Current address (J. Hejtmánek): Mathematisches Institut der Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria