

A COVERING LEMMA FOR PRODUCT SPACES

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Dedicated to Mischa Cotlar

ABSTRACT. We give a substitute for the Whitney decomposition of an arbitrary open set in \mathbf{R}^2 where squares are replaced by rectangles. Then we deduce the L^∞ -BMO boundedness of certain singular integral operators defined on product spaces.

In [2] it is shown that $\text{BMO}(\mathbf{R} \times \mathbf{R})$, the dual of $H^1(\mathbf{R} \times \mathbf{R})$, can be characterized in terms of Carleson measures on $\mathbf{R}_+^2 \times \mathbf{R}_+^2$ [1]. Let Ω be a bounded open set in \mathbf{R}^2 and $S(\Omega)$ be the shadow region over Ω , that is, the set of (x_1, x_2, t_1, t_2) such that $[x_1 - t_1, x_1 + t_1] \times [x_2 - t_2, x_2 + t_2] \subseteq \Omega$. Let $\psi \in C_0^\infty(\mathbf{R})$ be supported on $[-1, 1]$, such that $\int \psi dx = 0$, and let $Q_t = Q_{t_1} \otimes Q_{t_2}$ be the convolution with $(1/t_1 t_2)\psi(x_1/t_1)\psi(x_2/t_2)$. Then a function $b \in L_{\text{loc}}^2(\mathbf{R}^2)$ is in $\text{BMO}(\mathbf{R} \times \mathbf{R})$ if and only if there exists a constant C_b such that

$$(1) \quad \int_{S(\Omega)} |Q_t b|^2 dx_1 dx_2 \frac{dt_1}{t_1} \frac{dt_2}{t_2} \leq C_b^2 |\Omega|$$

and then $C_b \approx \|b\|_{\text{BMO}}$. The problem in proving (1) for a given function b is that one has to consider all possible bounded open sets and not simply rectangles. However, (1) can still be checked in special cases, using the following proposition.

PROPOSITION 1. *Let Ω be a bounded open subset of \mathbf{R}^2 , let $\{R_k, k \in K\}$ be the collection of maximal dyadic rectangles contained in Ω , and let $R_k = I_k \times J_k$ for all $k \in K$. There exist dyadic intervals $\{\tilde{I}_k, k \in K\}$ such that $I_k \subseteq \tilde{I}_k$ for all $k \in K$ and such that for all increasing functions $\omega: \{2^{-j}, j \in \mathbf{N}\} \rightarrow [0, +\infty)$*

$$(2) \quad \left\| \bigcup_{k \in K} \tilde{I}_k \times J_k \right\| \leq 2|\Omega|$$

and

$$(3) \quad \sum_{k \in K} |R_k| \omega \left(\frac{|I_k|}{|\tilde{I}_k|} \right) \leq 2 \left[\sum_{j=0}^{+\infty} \omega(2^{-j}) \right] |\Omega|.$$

An alternate way to state this proposition in nondyadic language is the following. For all $(x_1, t_1) \in \mathbf{R}_+^2$ let $E_{x_1 t_1}$ be the open set $\{x_2, [x_1 - t_1, x_1 + t_1] \times \{x_2\} \subseteq \Omega\}$ and $\bigcup_l I_{x_1 t_1}^l$ be its decomposition in connected components.

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PROPOSITION 2. *There exists a function $t_1^l(x_1, t_1)$ such that*

$$(4) \quad \left\| \bigcup_{x_1, t_1, l}]x_1 - t_1^l, x_1 + t_1^l[\times I_{x_1 t_1}^l \right\| \leq 8|\Omega|$$

and such that, for all $\omega: [0, 1] \rightarrow [0, +\infty)$ increasing,

$$(5) \quad \int_{x_1} \int_{t_1} \sum_l |I_{x_1 t_1}^l| \omega\left(\frac{t_1}{t_1^l}\right) \frac{dx_1 dt_1}{t_1} \leq 2 \left[\int_0^1 \frac{\omega(\delta)}{\delta} d\delta \right] |\Omega|.$$

Propositions 1 and 2 are formally the same and so are their proofs. Thus we shall only prove Proposition 2.

The function t_1^l is simply defined as

$$\inf\{t_1' \geq t_1, |E_{x_1 t_1'} \cap I_{x_1 t_1}^l| / |I_{x_1 t_1}^l| \leq 1/2\}.$$

To show (4) we observe that if $t_1' < t_1^l$, then $|E_{x_1 t_1'} \cap I_{x_1 t_1}^l| > \frac{1}{2}|I_{x_1 t_1}^l|$, and, therefore, since $]x_1 - t_1', x_1 + t_1'[\times E_{x_1 t_1'} \subseteq \Omega$,

$$|]x_1 - t_1', x_1 + t_1'[\times I_{x_1 t_1}^l \cap \Omega| > \frac{1}{2} \times 2t_1' \times |I_{x_1 t_1}^l|.$$

If we let Ω_1 be $\bigcup_{x_1, t_1, l}]x_1 - t_1^l, x_1 + t_1^l[\times I_{x_1 t_1}^l$ and if $*$ denotes the strong Hardy-Littlewood maximal function then $\frac{1}{2}\chi_{\Omega_1} \leq \chi_{\Omega}^*$, which implies (4).

To estimate the l.h.s of (5) we rewrite it as

$$(6) \quad \int_{x_1} \int_{t_1} \sum_l \int_{t_1' > t_1^l} |I_{x_1 t_1}^l| \omega'\left(\frac{t_1}{t_1'}\right) \frac{dt_1'}{t_1'^2} dt_1 dx_1,$$

assuming that ω is in $C^1[0, 1]$, which is no loss in generality. If $t_1' > t_1^l$, then $|E_{x_1 t_1'} \cap I_{x_1 t_1}^l| \leq \frac{1}{2}|I_{x_1 t_1}^l|$, so that $|I_{x_1 t_1}^l \setminus E_{x_1 t_1'}| \geq \frac{1}{2}|I_{x_1 t_1}^l|$. Hence, (6) is less than

$$(7) \quad \int_{x_1} \int_{t_1} \sum_l \int_{t_1' > t_1^l} \left[\int_{I_{x_1 t_1}^l \setminus E_{x_1 t_1'}} 2 dx_2 \right] \omega'\left(\frac{t_1}{t_1'}\right) \frac{dt_1'}{t_1'^2} dt_1 dx_1.$$

For $(x_1, x_2) \in \Omega$ fixed, let

$$T_1(x_1, x_2) = \sup\{s_1, [x_1 - s_1, x_1 + s_1] \times \{x_2\} \subseteq \Omega\}.$$

If $x_2 \in I_{x_1 t_1}^l \setminus E_{x_1 t_1'}$, then $t_1 \leq T_1(x_1, x_2) \leq t_1'$. Hence, (7) is less than

$$2 \iint_{(x_1, x_2) \in \Omega} \left[\int_{t_1 \leq T_1 \leq t_1'} \omega'\left(\frac{t_1}{t_1'}\right) \frac{dt_1'}{t_1'^2} \right] dx_1 dx_2,$$

which is exactly $2\left[\int_0^1 \omega(\delta) \frac{d\delta}{\delta}\right] \times |\Omega|$.

As we said, Proposition 1 is nothing but Proposition 2 reformulated in dyadic language. It also has the advantage of extending immediately to a product of type $\mathbf{R}^n \times \mathbf{R}^m$. For simplicity we stick to the case of $\mathbf{R} \times \mathbf{R}$ and consider the following problem. Let T_1 and T_2 be two Calderon-Zygmund operators as defined in [3] or [5]. Then $T_1 \otimes T_2$ is trivially bounded on $L^2(\mathbf{R}^2)$ and on $L^p(\mathbf{R}^2)$ for all $p \in]1, +\infty[$. Moreover, we have the following.

PROPOSITION 3. *Let T_1 and T_2 be two Calderon-Zygmund operators acting on $L^2(\mathbf{R})$. Then $T_1 \otimes T_2$ extends to an operator bounded from $L^\infty(\mathbf{R}^2)$ to $\text{BMO}(\mathbf{R} \times \mathbf{R})$.*

As is readily seen, the proof we give below requires only a very weak smoothness assumption on the kernels of T_1 and T_2 . Let T be a Calderon-Zygmund operator acting on $L^2(\mathbf{R})$. Then $Q_t T$ is well defined for all $t > 0$ and has a kernel $(Q_t T)(x, y)$, given when $|x - y| > t$ by

$$\int \psi_t(x - u)[K(u, y) - K(x, y)] du,$$

where K is the kernel of T . What we shall need is that for all $x \in \mathbf{R}$ and $t > 0$,

$$(8') \quad \int_{t_0 < |x-y|/2} |(Q_{t_0} T)(x, y)| dy < C(T)$$

and

$$(8'') \quad \iint_{t < t_0 < |x-y|/2} |(Q_t T)(x, y)| dy < \omega_T \left(\frac{t}{t_0} \right) \\ \text{with } \int_0^1 \omega_T(\delta) \frac{d\delta}{\delta} = C(T) < +\infty.$$

This will certainly be the case if the function ω , defined as

$$\omega(\delta) = \sup_{x, x'} \int_{|x-y| > 2|x-x'|/\delta} |K(x, y) - K(x', y)| dy,$$

satisfies $\omega(1) < +\infty$ and the Dini condition $\int_0^1 \omega(\delta)/\delta d\delta < +\infty$.

We should also remark that this proof extends with minor changes to the class of singular integral operators defined in [4] or [6].

To prove Proposition 3 we choose a bounded open set Ω in \mathbf{R}^2 , a function $b \in C_0^\infty(\mathbf{R}^2)$ such that $\|b\|_\infty < 1$, two Calderon-Zygmund operators T_1 and T_2 satisfying (8') and (8'') with a constant 1 and such that $\|T_1\|_{2,2} + \|T_2\|_{2,2} \leq 1$. Then we must show that

$$(9) \quad \int_{S(\Omega)} |Q_{t_1} Q_{t_2} S b|^2 dx_1 dx_2 \frac{dt_1}{t_1} \frac{dt_2}{t_2} \leq C|\Omega|$$

where S denotes $T_1 \otimes T_2$.

Recall that, by Plancherel's theorem, if $f \in L^2(\mathbf{R}^2)$,

$$\iint_{\mathbf{R}_+^2 \times \mathbf{R}_+^2} |Q_{t_1} Q_{t_2} f|^2 dx_1 dx_2 \frac{dt_1}{t_1} \frac{dt_2}{t_2} \leq C\|f\|_2^2.$$

Applying this to $f = S(b\chi_{\Omega_2})$, where $\Omega_2 = \bigcup_{R \subseteq \Omega_1} 2R$, and using

$$\|f\|_2^2 \leq \|S\|_{2,2}^2 \|b\|_\infty^2 |\Omega_2| \leq C|\Omega|,$$

we see that we may as well suppose that b is supported outside of Ω_2 .

For each $(x_2, t_2) \in \mathbf{R}_+^2$ let $F_{x_2 t_2} = \bigcup]x_1 - 2t_1^l, x_1 + 2t_1^l]$, where the union is over those (x_1, t_1, l) such that $]x_2 - t_2, x_2 + t_2] \subseteq I_{x_1 t_1}^l$. Observe that

$$\bigcup_{x_2, t_2} F_{x_2 t_2} \times]x_2 - 2t_2, x_2 + 2t_2] \subseteq \Omega_2.$$

We now split b as $b_{x_2 t_2}^1 + b_{x_2 t_2}^2$, as in [1], but in a different way. Here $b_{x_2 t_2}^1(z_1, z_2) = b(z_1, z_2)\chi_{F_{x_2 t_2}}(z_1)$, so that $b_{x_2 t_2}^1(z_1, z_2) = 0$ for all z_1 if $|z_2 - x_2| \leq 2t_2$, since b is supported out of Ω_2 .

In order to prove (9) with $b_{x_2 t_2}^1$ instead of b , observe that $[Q_{t_1} Q_{t_2} S b_{x_2 t_2}^1](x_1, x_2)$ can be written as

$$\int_{|x_2 - z_2| \geq 2t_2} (Q_{t_2} T_2)(x_2, z_2) [Q_{t_1} T_1 b_{x_2 t_2}^1(\cdot, z_2)](x_1) dz_2.$$

Minkowski's inequality implies that

$$\iint |(Q_{t_1} Q_{t_2} S b_{x_2 t_2}^1)(x_1, x_2)|^2 \frac{dx_1 dt_1}{t_1}$$

is less than

$$\left[\int_{|x_2 - z_2| \geq 2t_2} |(Q_{t_2} T_2)(x_2, z_2)| \left[\iint |[Q_{t_1} T_1 b_{x_2 t_2}^1(\cdot, z_2)](x_1)|^2 \frac{dx_1 dt_1}{t_1} \right]^{1/2} dz_2 \right]^2.$$

Using Cauchy-Schwarz, (8') and the L^2 -boundedness of T_1 , we see that this integral is dominated by

$$(10) \quad \iint_{(z_1, z_2) \in \text{supp } b_{x_2 t_2}^1} |(Q_{t_2} T_2)(x_2, z_2)| dz_1 dz_2.$$

We integrate (10) with respect to $dx_2 dt_2/t_2$ and fix x_2 and z_1 . Observe that $(z_1, x_2) \in \Omega_2$ if $z_1 \in F_{x_2 t_2}$ for some $t_2 \geq 0$. Therefore, to prove (9) with $b_{x_2 t_2}^1$ instead of b it is enough to prove

$$(11) \quad \iint_{(z_2, t_2) \in A_{x_2 z_1}} |(Q_{t_2} T_2)(x_2, z_2)| dz_2 \frac{dt_2}{t_2} \leq C,$$

where

$$A_{x_2 z_1} = \{(z_2, t_2), |x_2 - z_2| \geq 2t_2, (z_1, z_2) \in \text{supp } b_{x_2 t_2}^1\}.$$

Let $\tau(z_1, x_2) = \sup\{t_2, z_1 \in F_{x_2 t_2}\}$. If $(z_1, z_2) \in \text{supp } b_{x_2 t_2}^1$, then $t_2 \leq \tau(z_1, x_2)$. On the other hand,

$$\{z_1\} \times]x_2 - 2\tau, x_2 + 2\tau[\subseteq \bigcup_{t_2} F_{x_2 t_2} \times]x_2 - 2t_2, x_2 + 2t_2[\subseteq \Omega_2.$$

Therefore if $(z_1, z_2) \in \text{supp } b_{x_2 t_2}^1$, $(z_1, z_2) \in \Omega_2^c$ and $|x_2 - z_2| \geq 2\tau$, so that $(z_2, t_2) \in A_{x_2 z_1}$, then $t_2 \leq \tau \leq |x_2 - z_2|/2$. This, together with (8''), implies (11).

The proof of (9) with $b_{x_2 t_2}^2$ instead of b is along the same lines. As in [1] we rewrite the l.h.s. of (9) as

$$\iint_{(x_1, t_1)} \left(\sum_l \left[\int_{S(I_{x_1 t_1}^l)} |Q_{t_1} Q_{t_2} S b_{x_2 t_2}^2(x_1, x_2)|^2 \frac{dx_2 dt_2}{t_2} \right] \right) \frac{dx_1 dt_1}{t_1}.$$

Recall that $b_{x_2 t_2}^2(z_1, z_2) = b\chi_{F_{x_2 t_2}^c}(z_1)$. If $(x_2, t_2) \in S(I_{x_1 t_1}^l)$ and $z_1 \notin F_{x_2 t_2}$, then, by definition, $|x_1 - z_1| \geq 2t_1^l(x_1, t_1)$. Therefore, $b_{x_2 t_2}^2(z_1, z_2) = 0$ if $|z_1 - x_1| \leq 2t_1^l$. Now we develop in the variable z_1 as we did before in the variable z_2 , use

Minkowski's inequality in z_1 , the L^∞ -BMO boundedness of T_2 , and finally we are reduced to estimating

$$\iint_{(x_1 t_1)} \left[\sum_l \int_{|x_1 - z_1| \geq 2t_1^l} |I_{x_1 t_1}^l| |(Q_{t_1} T)_{x_1 z_1}| dz_1 \right] \frac{dx_1 dt_1}{t_1}.$$

By (8'') and (5) this is less than $C|\Omega|$. Proposition 3 is proved.

Recall that in one variable one associates to any function $\psi \in C_0^\infty(\mathbb{R})$ with $\int \psi dx = 0$ a Littlewood-Paley G -function defined by

$$Gf(x) = \left[\int_0^{+\infty} (\psi_t * f)^2(x) \frac{dt}{t} \right]^{1/2}.$$

By Plancherel's theorem $\|Gf\|_2 \leq C\|f\|_2$. Using vector-valued singular integral operators, we see also that $\|Gf\|_{\text{BMO}} \leq C\|f\|_{\text{BMO}}$. It is known that $\|(Gf)^2\|_{\text{BMO}} \leq C\|f\|_{\text{BMO}}^2$. We shall generalize this latter fact to the product situation.

PROPOSITION 4. Let $\psi \in C_0^\infty(\mathbb{R})$ with $\int \psi dx = 0$, and let Q_t be the convolution with $(1/t)\psi(\cdot/t)$. For all $f \in L_{\text{loc}}^2(\mathbb{R}^2)$, let

$$Gf(x) = \left[\int_0^{+\infty} \int_0^{+\infty} [(Q_{t_1} Q_{t_2} f)^2(x_1, x_2)]^2 \frac{dt_1 dt_2}{t_1 t_2} \right]^{1/2}.$$

Then $\|(Gf)^2\|_{\text{BMO}(\mathbb{R} \times \mathbb{R})} \leq C\|f\|_{\text{BMO}(\mathbb{R} \times \mathbb{R})}^2$.

To prove this proposition it is enough to show that the operator T_f defined for $f \in C_0^\infty(\mathbb{R}^2)$ by

$$T_f(g) = \int_0^{+\infty} \int_0^{+\infty} (Q_{t_1} Q_{t_2} f)(Q_{t_1} Q_{t_2} g) \frac{dt_1 dt_2}{t_1 t_2}$$

for g in $C_0^\infty(\mathbb{R}^2)$ extends to an operator bounded from $\text{BMO}(\mathbb{R} \times \mathbb{R})$ to itself with a norm at most $C\|f\|_{\text{BMO}}$. Actually it is rather easy to see that T_f is a CZO on $\mathbb{R} \times \mathbb{R}$ as defined in [6], to which the T1-theorem of [6] applies and also Proposition 3. Knowing that T_f maps L^∞ to BMO, it remains to show that $T_f H_1, T_f H_2, T_f H_1 H_2$, where H_1 and H_2 are the partial Hilbert transforms, also map L^∞ to BMO. Then the decomposition of BMO functions in terms of L^∞ and partial Hilbert transforms will imply that T_f maps BMO to itself. Now the fact that $T_f H_1, T_f H_2, T_f H_1 H_2$ are CZOs to which Proposition 3 applies follows easily from the fact that the kernel of $Q_t H$ has the smoothness and decay properties of the Poisson kernel p_t .

Another approach to the L^2 -boundedness of T_f uses the technique of [3, Chapter VI]. This is more powerful since one can also prove vector-valued inequalities. Let V be the Hilbert space

$$L^2 \left((\mathbb{R}_+)^2, \frac{dt_1 dt_2}{t_1 t_2} \right).$$

Let g be a V -valued $C_0^\infty(\mathbb{R}^2)$ function. Then $[\tilde{T}_f(g)](x)$ is defined as

$$\int_0^{+\infty} \int_0^{+\infty} [(Q_{t_1} Q_{t_2} f)(x)] [(Q_{t_1} Q_{t_2} g(\cdot, t_1, t_2))(x)] \frac{dt_1 dt_2}{t_1 t_2}.$$

Then \tilde{T}_f is also bounded from $L_V^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$ and from $\text{BMO}_V(\mathbb{R} \times \mathbb{R})$ to $\text{BMO}(\mathbb{R} \times \mathbb{R})$.

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