A COVERING LEMMA FOR PRODUCT SPACES

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Dedicated to Mischa Cotlar

ABSTRACT. We give a substitute for the Whitney decomposition of an arbitrary open set in $\mathbb{R}^2$ where squares are replaced by rectangles. Then we deduce the $L^\infty$-BMO boundedness of certain singular integral operators defined on product spaces.

In [2] it is shown that $\text{BMO}(\mathbb{R} \times \mathbb{R})$, the dual of $H^1(\mathbb{R} \times \mathbb{R})$, can be characterized in terms of Carleson measures on $\mathbb{R}^2_+ \times \mathbb{R}^2_+ [1]$. Let $\Omega$ be a bounded open set in $\mathbb{R}^2$ and $S(\Omega)$ be the shadow region over $\Omega$, that is, the set of $(x_1, x_2, t_1, t_2)$ such that $[x_1 - t_1, x_1 + t_1] \times [x_2 - t_2, x_2 + t_2] \subseteq \Omega$. Let $\psi \in C_0^\infty(\mathbb{R})$ be supported on $[-1, 1]$, such that $\int \psi \, dx = 0$, and let $Q_t = Q_{t_1} \otimes Q_{t_2}$ be the convolution with $(1/t_1 t_2) \psi(x_1/t_1) \psi(x_2/t_2)$. Then a function $b \in L^{2, \text{loc}}(\mathbb{R}^2)$ is in $\text{BMO}(\mathbb{R} \times \mathbb{R})$ if and only if there exists a constant $C_b$ such that

$$\int_{S(\Omega)} |Q_t b|^2 \, dx_1 \, dx_2 \, dt_1 \, dt_2 \leq C_b^2 |\Omega|$$

and then $C_b \approx ||b||_{\text{BMO}}$. The problem in proving (1) for a given function $b$ is that one has to consider all possible bounded open sets and not simply rectangles. However, (1) can still be checked in special cases, using the following proposition.

PROPOSITION 1. Let $\Omega$ be a bounded open subset of $\mathbb{R}^2$, let $\{R_k, k \in K\}$ be the collection of maximal dyadic rectangles contained in $\Omega$, and let $R_k = I_k \times J_k$ for all $k \in K$. There exist dyadic intervals $\{I_k, k \in K\}$ such that $I_k \subseteq I_k$ for all $k \in K$ and such that for all increasing functions $\omega$: $\{2^{-j}, j \in \mathbb{N}\} \rightarrow [0, +\infty)$

$$\left\| \bigcup_{k \in K} I_k \times J_k \right\| \leq 2|\Omega|$$

and

$$\sum_{k \in K} |R_k| \omega \left( \frac{|I_k|}{|I_k|} \right) \leq 2 \left[ \sum_{j=0}^{+\infty} \omega(2^{-j}) \right] |\Omega|.$$
PROPOSITION 2. There exists a function $t'_1(x_1,t_1)$ such that

$$
\left\| \bigcup_{x_1,t_1} [x_1 - t'_1, x_1 + t'_1] \times I_{x_1,t_1} \right\| \leq 8|\Omega|
$$

and such that, for all $\omega : [0,1] \to [0, +\infty)$ increasing,

$$
\int_{x_1} \int_{t_1} \sum_i |I_{x_1,t_1}| |\omega\left(\frac{t_1}{t'_1}\right) \frac{dt_1}{t_1}| \leq 2 \left[ \int_0^1 \frac{\omega(\delta)}{\delta} d\delta \right] |\Omega|.
$$

Propositions 1 and 2 are formally the same and so are their proofs. Thus we shall only prove Proposition 2.

The function $t'_1$ is simply defined as

$$
\inf\{t'_1 \geq t_1, |E_{x_1,t'_1} \cap I_{x_1,t_1}| / |I_{x_1,t_1}| \leq 1/2\}.
$$

To show (4) we observe that if $t'_1 < t'_1$, then $|E_{x_1,t'_1} \cap I_{x_1,t_1}| > \frac{1}{2} |I_{x_1,t_1}|$, and, therefore, since $|x_1 - t'_1, x_1 + t'_1] \times E_{x_1,t'_1} \subseteq \Omega$,

$$
|\{x_1 - t'_1, x_1 + t'_1] \times I_{x_1,t_1} \cap \Omega| > \frac{1}{2} \times 2t'_1 \times |I_{x_1,t_1}|.
$$

If we let $\Omega_1$ be $\bigcup_{x_1,t_1} [x_1 - t'_1, x_1 + t'_1] \times I_{x_1,t_1}$ and if $\ast$ denotes the strong Hardy-Littlewood maximal function then $\frac{1}{2} \chi_{\Omega_1} \leq \chi_{\Omega}$, which implies (4).

To estimate the l.h.s of (5) we rewrite it as

$$
\int_{x_1} \int_{t_1} \sum_i \int_{t'_1 > t'_1} |I_{x_1,t_1}| \omega\left(\frac{t_1}{t'_1}\right) \frac{dt_1}{t'_1} dt_1 dx_1,
$$

assuming that $\omega$ is in $C^1[0,1]$, which is no loss in generality. If $t'_1 > t'_1$, then $|E_{x_1,t'_1} \cap I_{x_1,t_1}| \leq \frac{1}{2} |I_{x_1,t_1}|$, so that $|I_{x_1,t_1} \setminus E_{x_1,t'_1}| \geq \frac{1}{2} |I_{x_1,t_1}|$. Hence, (6) is less than

$$
\int_{x_1} \int_{t_1} \sum_i \int_{t'_1 > t'_1} \int_{E_{x_1,t'_1}} 2 dx_2 \omega\left(\frac{t_1}{t'_1}\right) \frac{dt_1}{t'_1} dx_1.
$$

For $(x_1, x_2) \in \Omega$ fixed, let

$$
T_1(x_1, x_2) = \sup\{s_1, |x_1 - s_1, x_1 + s_1] \times \{x_2\} \subseteq \Omega\}.
$$

If $x_2 \in I_{x_1,t_1} \setminus E_{x_1,t'_1}$, then $t_1 \leq T_1(x_1, x_2) \leq t'_1$. Hence, (7) is less than

$$
2 \int \int_{(x_1, x_2) \in \Omega} \int_{t_1 \leq T_1 \leq t'_1} \omega\left(\frac{t_1}{t'_1}\right) \frac{dt_1}{t'_1} dx_1 dx_2,
$$

which is exactly $2\left[ \int_0^1 \omega(\delta) \frac{d\delta}{\delta} \right] \times |\Omega|$.

As we said, Proposition 1 is nothing but Proposition 2 reformulated in dyadic language. It also has the advantage of extending immediately to a product of type $\mathbb{R}^n \times \mathbb{R}^m$. For simplicity we stick to the case of $\mathbb{R} \times \mathbb{R}$ and consider the following problem. Let $T_1$ and $T_2$ be two Calderon-Zygmund operators as defined in [3] or [5]. Then $T_1 \otimes T_2$ is trivially bounded on $L^2(\mathbb{R}^2)$ and on $L^p(\mathbb{R}^2)$ for all $p \in ]1, +\infty[$. Moreover, we have the following.
PROPOSITION 3. Let $T_1$ and $T_2$ be two Calderon-Zygmund operators acting on $L^2(\mathbb{R})$. Then $T_1 \otimes T_2$ extends to an operator bounded from $L^\infty(\mathbb{R}^2)$ to $\text{BMO}(\mathbb{R} \times \mathbb{R})$.

As is readily seen, the proof we give below requires only a very weak smoothness assumption on the kernels of $T_1$ and $T_2$. Let $T$ be a Calderon-Zygmund operator acting on $L^2(\mathbb{R})$. Then $Q_tT$ is well defined for all $t > 0$ and has a kernel $(Q_tT)(x, y)$, given when $|x - y| > t$ by

$$
\int \psi_t(x - u)[K(u, y) - K(x, y)] \, du,
$$

where $K$ is the kernel of $T$. What we shall need is that for all $x \in \mathbb{R}$ and $t > 0$,

$$(8') \int_{|x-y|/2}^{t} |(Q_{t_0}T)(x, y)| \, dy < C(T)
$$

and

$$(8'') \iint_{t-t_0<|x-y|/2} |(Q_tT)(x, y)| \, dy < \omega_T \left( \frac{t}{t_0} \right)$$

with $\int_0^1 \omega_T(\delta) \frac{d\delta}{\delta} = C(T) < +\infty$.

This will certainly be the case if the function $\omega$, defined as

$$\omega(\delta) = \sup_{x, x'} \int_{|x-y|>2|x-x'|/\delta} |K(x, y) - K(x', y)| \, dy,$$

satisfies $\omega(1) < +\infty$ and the Dini condition $\int_0^1 \omega(\delta)/\delta \, d\delta < +\infty$.

We should also remark that this proof extends with minor changes to the class of singular integral operators defined in [4] or [6].

To prove Proposition 3 we choose a bounded open set $\Omega$ in $\mathbb{R}^2$, a function $b \in C_0^\infty(\mathbb{R}^2)$ such that $||b||_\infty < 1$, two Calderon-Zygmund operators $T_1$ and $T_2$ satisfying (8') and (8'') with a constant 1 and such that $||T_1||_{2,2} + ||T_2||_{2,2} \leq 1$. Then we must show that

$$(9) \int_{S(\Omega)} |Q_{t_1}Q_{t_2}Sb|^2 \, dx_1 \, dx_2 \frac{dt_1}{t_1} \frac{dt_2}{t_2} \leq C|\Omega|$$

where $S$ denotes $T_1 \otimes T_2$.

Recall that, by Plancherel's theorem, if $f \in L^2(\mathbb{R}^2)$,

$$\iint_{\mathbb{R}^2 \times \mathbb{R}^2} |Q_{t_1}Q_{t_2}f|^2 \, dx_1 \, dx_2 \frac{dt_1}{t_1} \frac{dt_2}{t_2} \leq C||f||_2^2.$$ 

Applying this to $f = S(b_{\chi_{\Omega_2}})$, where $\Omega_2 = \bigcup_{R \subseteq \Omega_1} 2R$, and using

$$||f||_2^2 \leq ||S||_{2,2} ||b||_\infty^2 |\Omega_2| \leq C|\Omega|,$$

we see that we may as well suppose that $b$ is supported outside of $\Omega_2$.

For each $(x_2, t_2) \in \mathbb{R}^2_+$ let $F_{x_2 t_2} = \bigcup_{x_1} [x_1 - 2t_1, x_1 + 2t_1]$, where the union is over those $(x_1, t_1, l)$ such that $[x_2 - t_2, x_2 + t_2] \subseteq I_{x_1 t_1}$. Observe that

$$\bigcup_{x_2, t_2} F_{x_2 t_2} \times [x_2 - 2t_2, x_2 + 2t_2] \subseteq \Omega_2.$$
We now split $b$ as $b_{x,t_2}^1 + b_{x,t_2}^2$, as in [1], but in a different way. Here $b_{x,t_2}^1(z_1, z_2) = b(z_1, z_2) \chi_{F_{x,t_2}}(z_2)$, so that $b^1_{x,t_2}(z_1, z_2) = 0$ for all $z_1$ if $|z_2 - z_1| \leq 2t_2$, since $b$ is supported out of $\Omega_2$.

In order to prove (9) with $b_{x,t_2}^1$ instead of $b$, observe that $[Q_{t_1} Q_{t_2} S b_{x,t_2}^1](x_1, x_2)$ can be written as

$$\int_{|x_2 - z_2| \geq 2t_2} (Q_{t_2} T_2)(x_2, z_2) [Q_{t_1} T_1 b_{x,t_2}^1(\cdot, z_2)](x_1) \, dz_2.$$ 

Minkowski's inequality implies that

$$\int \int |(Q_{t_1} Q_{t_2} S b_{x,t_2}^1)(x_1, x_2)|^2 \frac{dx_1 dt_1}{t_1}$$

is less than

$$\left[ \int_{|x_2 - z_2| \geq 2t_2} |(Q_{t_2} T_2)(x_2, z_2)| \left( \int \int |Q_{t_1} T_1 b_{x,t_2}^1(\cdot, z_2)](x_1)|^2 \frac{dx_1 dt_1}{t_1} \right)^{1/2} \, dz_2 \right]^2.$$ 

Using Cauchy-Schwarz, $(8')$ and the $L^2$-boundedness of $T_1$, we see that this integral is dominated by

$$\int \int \int_{(z_1, z_2) \in \text{supp } b_{x,t_2}^1} |(Q_{t_2} T_2)(x_2, z_2)| \, dz_1 \, dz_2.$$ 

We integrate (10) with respect to $dx_2 \, dt_2 / t_2$ and fix $x_2$ and $z_2$. Observe that $(z_1, x_2) \in \Omega_2$ if $z_1 \in F_{x,t_2}$ for some $t_2 \geq 0$. Therefore, to prove (9) with $b_{x,t_2}^1$ instead of $b$ it is enough to prove

$$\int \int \int_{(z_2, t_2) \in A_{x_2, z_1}} |(Q_{t_2} T_2)(x_2, z_2)| \, dz_2 \frac{dt_2}{t_2} \leq C,$$

where

$$A_{x_2, z_1} = \{(z_2, t_2), |x_2 - z_2| \geq 2t_2, (z_1, z_2) \in \text{supp } b_{x,t_2}^1 \}. $$

Let $\tau(z_1, x_2) = \sup \{t_2, z_1 \in F_{x,t_2} \}$. If $(z_1, z_2) \in \text{supp } b_{x,t_2}^1$, then $t_2 \leq \tau(z_1, x_2)$. On the other hand,

$$\{z_1\} \times [x_2 - 2\tau, x_2 + 2\tau] \subseteq \bigcup_{t_2} F_{x,t_2} \times [x_2 - 2t_2, x_2 + 2t_2] \subseteq \Omega_2.$$ 

Therefore if $(z_1, z_2) \in \text{supp } b_{x,t_2}^1$, $(z_2, t_2) \in A_{x_2, z_1}$ and $|x_2 - z_2| \geq 2\tau$, so that if $(z_2, t_2) \in A_{x_2, z_1}$, then $t_2 \leq \tau \leq |x_2 - z_2| / 2$. This, together with $(8'\prime)$, implies (11).

The proof of (9) with $b_{x,t_2}^2$ instead of $b$ is along the same lines. As in [1] we rewrite the l.h.s. of (9) as

$$\int \int \left( \sum_t \left[ \int_{S(I_{x_1, t_1})} |Q_{t_1} Q_{t_2} S b_{x,t_2}^2(x_1, x_2)|^2 \frac{dx_2 \, dt_2}{t_2} \right] \right) \frac{dx_1 \, dt_1}{t_1}.$$ 

Recall that $b_{x,t_2}^2(z_1, z_2) = b \chi_{F_{x,t_2}}(z_1)$. If $(x_2, t_2) \in S(I_{x_1, t_1})$ and $z_1 \not\in F_{x,t_2}$, then, by definition, $|z_1 - z_1| \geq 2t_1(x_1, t_1)$. Therefore, $b_{x,t_2}^2(z_1, z_2) = 0$ if $|z_1 - x_1| \leq 2t_1$. Now we develop in the variable $z_1$ as we did before in the variable $z_2$, use
Minkowski’s inequality in \( z_1 \), the \( L^\infty \)-BMO boundedness of \( T_2 \), and finally we are reduced to estimating
\[
\int \int_{(x_1,t_1)} \left[ \sum_{l} \int_{|x_1 - x_2| \geq 2t_1} |I_{x_1,t_1}^l ||(Q_{t_1}T)_{x_1,x_2}| \right] \frac{dx_1 dt_1}{t_1}.
\]
By \((8'')\) and \((5)\) this is less than \( C|\Omega| \). Proposition 3 is proved.

Recall that in one variable one associates to any function \( \psi \in \mathcal{C}_0^\infty (R) \) with \( \int \psi \, dx = 0 \) a Littlewood-Paley \( G \)-function defined by
\[
Gf(x) = \left[ \int_{0}^{+\infty} (\psi_t * f)^2(x) \frac{dt}{t} \right]^{1/2}.
\]
By Plancherel’s theorem \( ||Gf||_2 \leq C||f||_2 \). Using vector-valued singular integral operators, we see also that \( ||Gf||_{\text{BMO}} \leq C||f||_{\text{BMO}} \). It is known that \( \|(Gf)^2\|_{\text{BMO}} < C||f||_{\text{BMO}}^2 \). We shall generalize this latter fact to the product situation.

**Proposition 4.** Let \( \psi \in \mathcal{C}_0^\infty (R) \) with \( \int \psi \, dx = 0 \), and let \( Q_t \) be the convolution with \((1/t)\psi(-/t)\). For all \( f \in L^2_{\text{loc}}(R^2) \), let
\[
G f(x) = \left[ \int_{0}^{+\infty} \int_{0}^{+\infty} [(Q_{t_1}Q_{t_2}f)(x_1,x_2)]^2 \frac{dt_1 dt_2}{t_1 t_2} \right]^{1/2}.
\]
Then \( ||(Gf)^2||_{\text{BMO}(R \times R)} \leq C||f||^2_{\text{BMO}(R \times R)}. \)

To prove this proposition it is enough to show that the operator \( T_f \) defined for \( f \in \mathcal{C}_0^\infty (R^2) \) by
\[
T_f(g) = \int_{0}^{+\infty} \int_{0}^{+\infty} (Q_{t_1}Q_{t_2}f)(Q_{t_1}Q_{t_2}g) \frac{dt_1 dt_2}{t_1 t_2}
\]
for \( g \in \mathcal{C}_0^\infty (R^2) \) extends to an operator bounded from \( \text{BMO}(R \times R) \) to itself with a norm at most \( C||f||_{\text{BMO}} \). Actually it is rather easy to see that \( T_f \) is a CZO on \( R \times R \) as defined in \([6]\), to which the \( T_1 \)-theorem of \([6]\) applies and also Proposition 3. Knowing that \( T_f \) maps \( L^\infty \) to \( \text{BMO} \), it remains to show that \( T_f \) maps \( \mathcal{H}_1, \mathcal{H}_2, T_f \mathcal{H}_1 H_2 \), where \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are the partial Hilbert transforms, also map \( L^\infty \) to \( \text{BMO} \). Then the decomposition of \( \text{BMO} \) functions in terms of \( L^\infty \) and partial Hilbert transforms will imply that \( T_f \) maps \( \text{BMO} \) to itself. Now the fact that \( T_f \mathcal{H}_1, T_f \mathcal{H}_2, T_f H_1 H_2 \) are CZOs to which Proposition 3 applies follows easily from the fact that the kernel of \( Q_t H \) has the smoothness and decay properties of the Poisson kernel \( p_t \).

Another approach to the \( L^2 \)-boundedness of \( T_f \) uses the technique of \([3, \text{Chapter VI}]\). This is more powerful since one can also prove vector-valued inequalities. Let \( V \) be the Hilbert space
\[
L^2 \left( \mathbb{R}_+^2, \frac{dt_1 dt_2}{t_1 t_2} \right).
\]
Let \( g \) be a \( V \)-valued \( \mathcal{C}_0^\infty (R^2) \) function. Then \( \tilde{T}_f(g)(x) \) is defined as
\[
\int_{0}^{+\infty} \int_{0}^{+\infty} [(Q_{t_1}Q_{t_2}f)(Q_{t_1}Q_{t_2}g)(x)] \frac{dt_1 dt_2}{t_1 t_2}.
\]
Then \( \tilde{T}_f \) is also bounded from \( L^2_V (R^2) \) to \( L^2(R^2) \) and from \( \text{BMO}_V (R \times R) \) to \( \text{BMO}(R \times R) \).

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