PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 96, Number 4, April 1986

A COVERING LEMMA FOR PRODUCT SPACES

JEAN-LIN JOURNÉ

Dedicated to Mischa Cotlar

ABSTRACT. We give a substitute for the Whitney decomposition of an arbitrary open set in $\mathbf{R}^2$ where squares are replaced by rectangles. Then we deduce the L^∞-BMO boundedness of certain singular integral operators defined on product spaces.

In [**2**] it is shown that BMO($\mathbf{R}\times\mathbf{R}$), the dual of $H^1(\mathbf{R}\times\mathbf{R})$, can be characterized in terms of Carleson measures on $\mathbf{R}_+^2 \times \mathbf{R}_+^2$ [**1**]. Let Ω be a bounded open set in $\mathbf{R}^2$ and $S(\Omega)$ be the shadow region over Ω, that is, the set of (x_1, x_2, t_1, t_2) such that $[x_1 - t_1, x_1 + t_1] \times [x_2 - t_2, x_2 + t_2] \subseteq \Omega$. Let $\psi \in C_0^\infty(\mathbf{R})$ be supported on $[-1,1]$, such that $\int \psi\, dx = 0$, and let $Q_t = Q_{t_1} \otimes Q_{t_2}$ be the convolution with $(1/t_1t_2)\psi(x_1/t_1)\psi(x_2/t_2)$. Then a function $b \in L^2_{\mathrm{loc}}(\mathbf{R}^2)$ is in BMO($\mathbf{R}\times\mathbf{R}$) if and only if there exists a constant C_b such that

$$\int_{S(\Omega)} |Q_t b|^2\, dx_1\, dx_2 \frac{dt_1}{t_1}\frac{dt_2}{t_2} \leq C_b^2 |\Omega| \tag{1}$$

and then $C_b \approx ||b||_{\mathrm{BMO}}$. The problem in proving (1) for a given function b is that one has to consider all possible bounded open sets and not simply rectangles. However, (1) can still be checked in special cases, using the following proposition.

PROPOSITION 1. *Let Ω be a bounded open subset of $\mathbf{R}^2$, let $\{R_k, k \in K\}$ be the collection of maximal dyadic rectangles contained in Ω, and let $R_k = I_k \times J_k$ for all $k \in K$. There exist dyadic intervals $\{\tilde{I}_k, k \in K\}$ such that $I_k \subseteq \tilde{I}_k$ for all $k \in K$ and such that for all increasing functions $\omega\colon \{2^{-j}, j \in N\} \to [0, +\infty)$*

$$\left\| \bigcup_{k\in K} \tilde{I}_k \times J_k \right\| \leq 2|\Omega| \tag{2}$$

and

$$\sum_{k\in K} |R_k| \omega\left(\frac{|I_k|}{|\tilde{I}_k|}\right) \leq 2\left[\sum_{j=0}^{+\infty} \omega(2^{-j})\right] |\Omega|. \tag{3}$$

An alternate way to state this proposition in nondyadic language is the following. For all $(x_1, t_1) \in R_+^2$ let $E_{x_1t_1}$ be the open set $\{x_2, [x_1 - t_1, x_1 + t_1] \times \{x_2\} \subseteq \Omega\}$ and $\bigcup_l I^l_{x_1t_1}$ be its decomposition in connected components.

Received by the editors November 27, 1984.
1980 *Mathematics Subject Classification*. Primary 42-00.
Key words and phrases. Carleson measures, Whitney decompostion.

PROPOSITION 2. *There exists a function $t_1^l(x_1,t_1)$ such that*

$$\left\| \bigcup_{x_1,t_1,l}]x_1 - t_1^l, x_1 + t_1^l[\ \times I^l_{x_1t_1} \right\| \leq 8|\Omega| \tag{4}$$

and such that, for all $\omega\colon [0,1] \to [0,+\infty)$ increasing,

$$\int_{x_1}\int_{t_1}\sum_l |I^l_{x_1t_1}|\omega\left(\frac{t_1}{t_1^l}\right)\frac{dx_1dt_1}{t_1} \leq 2\left[\int_0^1 \frac{\omega(\delta)}{\delta}d\delta\right]|\Omega|. \tag{5}$$

Propositions 1 and 2 are formally the same and so are their proofs. Thus we shall only prove Proposition 2.

The function t_1^l is simply defined as

$$\inf\{t_1' \geq t_1, |\ E_{x_1t_1'} \cap I^l_{x_1t_1}\ | / |I^l_{x_1t_1}| \leq 1/2\}.$$

To show (4) we observe that if $t_1' < t_1^l$, then $|E_{x_1t_1'} \cap I^l_{x_1t_1}| > \frac{1}{2}|I^l_{x_1t_1}|$, and, therefore, since $]x_1 - t_1', x_1 + t_1'[\ \times E_{x_1t_1'} \subseteq \Omega$,

$$|\,]x_1 - t_1', x_1 + t_1'[\times I^l_{x_1t_1} \cap \Omega| > \tfrac{1}{2} \times 2t_1' \times |I^l_{x_1t_1}|.$$

If we let Ω_1 be $\bigcup_{x_1,t_1,l}]x_1 - t_1^l, x_1 + t_1^l[\times I^l_{x_1t_1}$ and if $*$ denotes the strong Hardy-Littlewood maximal function then $\frac{1}{2}\chi_{\Omega_1} \leq \chi^*_\Omega$, which implies (4).

To estimate the l.h.s of (5) we rewrite it as

$$\int_{x_1}\int_{t_1}\sum_l\int_{t_1'>t_1^l}|I^l_{x_1t_1}|\omega'\left(\frac{t_1}{t_1'}\right)\frac{dt_1'}{t_1'^2}\,dt_1\,dx_1, \tag{6}$$

assuming that ω is in $C^1[0,1]$, which is no loss in generality. If $t_1' > t_1^l$, then $|E_{x_1t_1'} \cap I^l_{x_1t_1}| \leq \frac{1}{2}|I^l_{x_1t_1}|$, so that $|I^l_{x_1t_1} \setminus E_{x_1t_1'}| \geq \frac{1}{2}|I^l_{x_1t_1}|$. Hence, (6) is less than

$$\int_{x_1}\int_{t_1}\sum_l\int_{t_1'>t_1^l}\left[\int_{I^l_{x_1t_1}\setminus E_{x_1t_1'}} 2\,dx_2\right]\omega'\left(\frac{t_1}{t_1'}\right)\frac{dt_1'}{t_1'^2}dt_1\,dx_1. \tag{7}$$

For $(x_1,x_2) \in \Omega$ fixed, let

$$T_1(x_1,x_2) = \sup\{s_1, [x_1 - s_1, x_1 + s_1] \times \{x_2\} \subseteq \Omega\}.$$

If $x_2 \in I^l_{x_1t_1} \setminus E_{x_1t_1'}$, then $t_1 \leq T_1(x_1,x_2) \leq t_1'$. Hence, (7) is less than

$$2\iint_{(x_1,x_2)\in\Omega}\left[\int_{t_1\leq T_1\leq t_1'}\omega'\left(\frac{t_1}{t_1'}\right)\frac{dt_1'}{t_1'^2}\right]dx_1\,dx_2,$$

which is exactly $2[\int_0^1\omega(\delta)\frac{d\delta}{\delta}] \times |\Omega|$.

As we said, Proposition 1 is nothing but Proposition 2 reformulated in dyadic language. It also has the advantage of extending immediately to a product of type $\mathbf{R}^n \times \mathbf{R}^m$. For simplicity we stick to the case of $\mathbf{R} \times \mathbf{R}$ and consider the following problem. Let T_1 and T_2 be two Calderon-Zygmund operators as defined in [**3**] or [**5**]. Then $T_1 \otimes T_2$ is trivially bounded on $L^2(\mathbf{R}^2)$ and on $L^p(\mathbf{R}^2)$ for all $p \in]1,+\infty[$. Moreover, we have the following.

PROPOSITION 3. *Let T_1 and T_2 be two Calderon-Zygmund operators acting on $L^2(\mathbf{R})$. Then $T_1 \otimes T_2$ extends to an operator bounded from $L^\infty(\mathbf{R}^2)$ to* $\mathrm{BMO}(\mathbf{R}\times\mathbf{R})$.

As is readily seen, the proof we give below requires only a very weak smoothness assumption on the kernels of T_1 and T_2. Let T be a Calderon-Zygmund operator acting on $L^2(\mathbf{R})$. Then $Q_t T$ is well defined for all $t > 0$ and has a kernel $(Q_t T)(x, y)$, given when $|x - y| > t$ by

$$\int \psi_t(x-u)[K(u,y) - K(x,y)]\, du,$$

where K is the kernel of T. What we shall need is that for all $x \in \mathbf{R}$ and $t > 0$,

$$\int_{t_0<|x-y|/2} |(Q_{t_0}T)(x,y)|\, dy < C(T) \tag{8$'$}$$

and

$$\iint_{t<t_0<|x-y|/2} |(Q_t T)(x,y)| dy < \omega_T\left(\frac{t}{t_0}\right) \quad \text{with } \int_0^1 \omega_T(\delta)\frac{d\delta}{\delta} = C(T) < +\infty. \tag{8$''$}$$

This will certainly be the case if the function ω, defined as

$$\omega(\delta) = \sup_{x,x'} \int_{|x-y|>2|x-x'|/\delta} |K(x,y) - K(x',y)|\, dy,$$

satisfies $\omega(1) < +\infty$ and the Dini condition $\int_0^1 \omega(\delta)/\delta\, d\delta < +\infty$.

We should also remark that this proof extends with minor changes to the class of singular integral operators defined in [**4**] or [**6**].

To prove Proposition 3 we choose a bounded open set Ω in $\mathbf{R}^2$, a function $b \in C_0^\infty(\mathbf{R}^2)$ such that $||b||_\infty < 1$, two Calderon-Zygmund operators T_1 and T_2 satisfying (8$'$) and (8$''$) with a constant 1 and such that $||T_1||_{2,2} + ||T_2||_{2,2} \le 1$. Then we must show that

$$\int_{S(\Omega)} |Q_{t_1}Q_{t_2}Sb|^2\, dx_1\, dx_2 \frac{dt_1}{t_1}\frac{dt_2}{t_2} \le C|\Omega| \tag{9}$$

where S denotes $T_1 \otimes T_2$.

Recall that, by Plancherel's theorem, if $f \in L^2(\mathbf{R}^2)$,

$$\iint_{\mathbf{R}^2_+\times\mathbf{R}^2_+} |Q_{t_1}Q_{t_2}f|^2\, dx_1\, dx_2 \frac{dt_1}{t_1}\frac{dt_2}{t_2} \le C||f||_2^2.$$

Applying this to $f = S(b\chi_{\Omega_2})$, where $\Omega_2 = \bigcup_{R\subseteq\Omega_1} 2R$, and using

$$||f||_2^2 \le ||S||_{2,2}^2 ||b||_\infty^2 |\Omega_2| \le C|\Omega|,$$

we see that we may as well suppose that b is supported outside of Ω_2.

For each $(x_2, t_2) \in \mathbf{R}^2_+$ let $F_{x_2t_2} = \bigcup]x_1 - 2t_1^l, x_1 + 2t_1^l]$, where the union is over those (x_1, t_1, l) such that $[x_2 - t_2, x_2 + t_2] \subseteq I^l_{x_1t_1}$. Observe that

$$\bigcup_{x_2,t_2} F_{x_2t_2} \times\ ]x_2 - 2t_2, x_2 + 2t_2[\subseteq \Omega_2.$$

We now split b as $b^1_{x_2t_2} + b^2_{x_2t_2}$, as in [**1**], but in a different way. Here $b^1_{x_2t_2}(z_1,z_2) = b(z_1,z_2)\chi_{F_{x_2t_2}}(z_1)$, so that $b^1_{x_2t_2}(z_1,z_2) = 0$ for all z_1 if $|z_2 - x_2| \le 2t_2$, since b is supported out of Ω_2.

In order to prove (9) with $b^1_{x_tt_2}$ instead of b, observe that $[Q_{t_1}Q_{t_2}Sb^1_{x_2t_2}](x_1,x_2)$ can be written as

$$\int_{|x_2-z_2|\ge 2t_2} (Q_{t_2}T_2)(x_2,z_2)[Q_{t_1}T_1b^1_{x_2t_2}(\cdot,z_2)](x_1)\,dz_2.$$

Minkowski's inequality implies that

$$\iint |(Q_{t_1}Q_{t_2}Sb^1_{x_2t_2})(x_1,x_2)|^2\frac{dx_1\,dt_1}{t_1}$$

is less than

$$\left[\int_{|x_2-z_2|\ge 2t_2} |(Q_{t_2}T_2)(x_2,z_2)| \left[\iint |[Q_{t_1}T_1b^1_{x_2t_2}(\cdot,z_2)](x_1)|^2\frac{dx_1\,dt_1}{t_1}\right]^{1/2} dz_2\right]^2.$$

Using Cauchy-Schwarz, (8′) and the L^2-boundedness of T_1, we see that this integral is dominated by

$$\iint_{(z_1,z_2)\in \operatorname{supp} b^1_{x_2t_2}} |(Q_{t_2}T_2)(x_2,z_2)|\,dz_1\,dz_2. \tag{10}$$

We integrate (10) with respect to $dx_2\,dt_2/t_2$ and fix x_2 and z_1. Observe that $(z_1,x_2)\in\Omega_2$ if $z_1\in F_{x_2t_2}$ for some $t_2\ge 0$. Therefore, to prove (9) with $b^1_{x_2t_2}$ instead of b it is enough to prove

$$\iint_{(z_2,t_2)\in A_{x_2z_1}} |(Q_{t_2}T_2)(x_2,z_2)|\,dz_2\,\frac{dt_2}{t_2} \le C, \tag{11}$$

where

$$A_{x_2z_1} = \{(z_2,t_2),\ |x_2 - z_2| \ge 2t_2,\ (z_1,z_2)\in \operatorname{supp} b^1_{x_2t_2}\}.$$

Let $\tau(z_1,x_2) = \sup\{t_2, z_1\in F_{x_2t_2}\}$. If $(z_1,z_2)\in\operatorname{supp} b^1_{x_2t_2}$, then $t_2\le\tau(z_1,x_2)$. On the other hand,

$$\{z_1\}\times\ ]x_2 - 2\tau, x_2+2\tau[\subseteq \bigcup_{t_2} F_{x_2t_2}\times\ ]x_2-2t_2, x_2+2t_2[\subseteq\Omega_2.$$

Therefore if $(z_1,z_2)\in\operatorname{supp} b^1_{x_2t_2}$, $(z_1,z_2)\in\Omega_2^c$ and $|x_2 - z_2|\ge 2\tau$, so that if $(z_2,t_2)\in A_{x_2z_1}$, then $t_2\le\tau\le|x_2 - z_2|/2$. This, together with (8″), implies (11).

The proof of (9) with $b^2_{x_2t_2}$ instead of b is along the same lines. As in [**1**] we rewrite the l.h.s. of (9) as

$$\iint_{(x_1,t_1)}\left(\sum_l\left[\int_{S(I^l_{x_1t_1})} |Q_{t_1}Q_{t_2}Sb^2_{x_2t_2}(x_1,x_2)|^2\frac{dx_2\,dt_2}{t_2}\right]\right)\frac{dx_1\,dt_1}{t_1}.$$

Recall that $b^2_{x_2t_2}(z_1,z_2) = b\chi_{F^c_{x_2t_2}}(z_1)$. If $(x_2,t_2)\in S(I^l_{x_1t_1})$ and $z_1\notin F_{x_2t_2}$, then, by definition, $|x_1 - z_1|\ge 2t^l_1(x_1,t_1)$. Therefore, $b^2_{x_2t_2}(z_1,z_2) = 0$ if $|z_1 - x_1|\le 2t^l_1$. Now we develop in the variable z_1 as we did before in the variable z_2, use

Minkowski's inequality in z_1, the L^∞-BMO boundedness of T_2, and finally we are reduced to estimating

$$\iint_{(x_1 t_1)} \left[\sum_l \int_{|x_1 - z_1| \geq 2t_1^l} |I^l_{x_1 t_1}|\,|(Q_{t_1}T)_{x_1 z_1}|\,dz_1\right] \frac{dx_1\,dt_1}{t_1}.$$

By (8″) and (5) this is less than $C|\Omega|$. Proposition 3 is proved.

Recall that in one variable one associates to any function $\psi \in C_0^\infty(R)$ with $\int \psi\,dx = 0$ a Littlewood-Paley G-function defined by

$$Gf(x) = \left[\int_0^{+\infty} (\psi_t * f)^2(x)\frac{dt}{t}\right]^{1/2}.$$

By Plancherel's theorem $||Gf||_2 \leq C||f||_2$. Using vector-valued singular integral operators, we see also that $||Gf||_{\mathrm{BMO}} \leq C||f||_{\mathrm{BMO}}$. It is known that $||(Gf)^2||_{\mathrm{BMO}} \leq C||f||^2_{\mathrm{BMO}}$. We shall generalize this latter fact to the product situation.

PROPOSITION 4. *Let $\psi \in C_0^\infty(R)$ with $\int \psi\,dx = 0$, and let Q_t be the convolution with $(1/t)\psi(\cdot/t)$. For all $f \in L^2_{\mathrm{loc}}(\mathbf{R}^2)$, let*

$$Gf(x) = \left[\int_0^{+\infty}\int_0^{+\infty} [(Q_{t_1}Q_{t_2}f)^2(x_1,x_2)]^2 \frac{dt_1\,dt_2}{t_1 t_2}\right]^{1/2}.$$

Then $||(Gf)^2||_{\mathrm{BMO}(\mathbf{R}\times\mathbf{R})} \leq C||f||^2_{\mathrm{BMO}(\mathbf{R}\times\mathbf{R})}$.

To prove this proposition it is enough to show that the operator T_f defined for $f \in C_0^\infty(\mathbf{R}^2)$ by

$$T_f(g) = \int_0^{+\infty}\int_0^{+\infty} (Q_{t_1}Q_{t_2}f)(Q_{t_1}Q_{t_2}g)\frac{dt_1}{t_1}\frac{dt_2}{t_2}$$

for g in $C_0^\infty(\mathbf{R}^2)$ extends to an operator bounded from $\mathrm{BMO}(\mathbf{R}\times\mathbf{R})$ to itself with a norm at most $C||f||_{\mathrm{BMO}}$. Actually it is rather easy to see that T_f is a CZO on $\mathbf{R}\times\mathbf{R}$ as defined in [**6**], to which the T1-theorem of [**6**] applies and also Proposition 3. Knowing that T_f maps L^∞ to BMO, it remains to show that $T_fH_1, T_fH_2, T_fH_1H_2$, where H_1 and H_2 are the partial Hilbert transforms, also map L^∞ to BMO. Then the decomposition of BMO functions in terms of L^∞ and partial Hilbert transforms will imply that T_f maps BMO to itself. Now the fact that $T_fH_1, T_fH_2, T_fH_1H_2$ are CZOs to which Proposition 3 applies follows easily from the fact that the kernel of Q_tH has the smoothness and decay properties of the Poisson kernel p_t.

Another approach to the L^2-boundedness of T_f uses the technique of [**3**, Chapter VI]. This is more powerful since one can also prove vector-valued inequalities. Let V be the Hilbert space

$$L^2\left((\mathbf{R}_+)^2, \frac{dt_1}{t_1}\frac{dt_2}{t_2}\right).$$

Let g be a V-valued $C_0^\infty(\mathbf{R}^2)$ function. Then $[\tilde{T}_f(g)](x)$ is defined as

$$\int_0^{+\infty}\int_0^{+\infty} [(Q_{t_1}Q_{t_2}f)(x)][(Q_{t_1}Q_{t_2}g(\cdot,t_1,t_2))(x)]\frac{dt_1}{t_1}\frac{dt_2}{t_2}.$$

Then $\tilde{T}_f$ is also bounded from $L^2_V(\mathbf{R}^2)$ to $L^2(\mathbf{R}^2)$ and from $\mathrm{BMO}_V(\mathbf{R}\times\mathbf{R})$ to $\mathrm{BMO}(\mathbf{R}\times\mathbf{R})$.

ACKNOWLEDGMENT. I would like to thank Guy David, Stephen Semmes and Fernando Soria for reading this manuscript and suggesting numerous improvements.

REFERENCES

1. S.-Y. A. Chang, *Carleson measure on the bi-disc*, Ann. of Math. (2) **109** (1979), 613–620.
2. S.-Y. A. Chang and R. Fefferman, *A continuous version of duality of H^1 with* BMO *on the bi-disc*, Ann. of Math. (2) **112** (1980), 179–201.
3. R. R. Coifman and Y. Meyer, *Au delà des opérateurs pseudo-differentiels*, Astérisque No. 57, 1978.
4. R. Fefferman and E. M. Stein, *Singular integrals on product spaces*, Adv. in Math. **45** (1982), 117–143.
5. J.-L. Journé, *Calderón-Zygmund operators, pseudo-differential operators and the Cauchy integral of Calderón*, Lecture Notes in Math., vol. 994, Springer-Verlag, Berlin and New York, 1983.
6. ———, *Calderón-Zygmund operators on product spaces*, Revista Ibero-Americana **3** (1985).

DEPARTMENT OF MATHEMATICS, YALE UNIVERSITY, NEW HAVEN, CONNECTICUT 06520

Current address: Department of Mathematics, Princeton University, Princeton, New Jersey 08544