PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 96, Number 4, April 1986

COINCIDENCE THEOREM AND SADDLE POINT THEOREM

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ABSTRACT. We discuss Browder's coincidence theorem and derive a saddle point theorem from it.

We always assume the Hausdorff separation axiom in topological structures. Let X be a topological space and let Y be a nonempty subset of a linear topological space F. By a *multi-valued mapping* A of X into Y, we mean that to each point x of X, A assigns a subset $A(x)$ of Y. A multi-valued mapping A is said to be *convex-valued* (resp. *closed convex-valued*) if $A(x)$ is nonempty and convex (resp. nonempty, closed and convex) for each x in X. A multi-valued mapping A is said to be *upper semicontinuous* if for each point x of X and each zero-neighborhood V of F, there exists a neighborhood U of x such that $A(u) \subset A(x) + V$ for all u in U. The upper semicontinuity of A is equivalent to the closedness of the graph $\mathrm{Gr}(A) = \{(x,y) \in X \times Y : y \in A(x)\}$ of A in $X \times Y$, if A is closed-valued and Y is compact.

There exist two fundamental fixed point theorems for multi-valued mappings. One is Kakutani-Fan's fixed point theorem:

Let X be a nonempty compact convex subset of a locally convex linear topological space. Let A be an upper semicontinuous and closed convex-valued mapping of X into X. Then A has a fixed point, that is, a point x_0 of X such that $x_0 \in A(x_0)$.

The other is Fan-Browder's fixed point theorem:

Let X be a nonempty compact convex subset of a linear topological space. Let B be a convex-valued mapping of X into X such that $B^{-1}(y) = \{x \in X : y \in Bx\}$ is open in X for each y in Y. Then B has a fixed point.

Kakutani and Fan's theorem has been established by Kakutani [**8**] in case X is contained in a finite dimensional space, and Fan [**4**] has generalized it to the present form. Fan and Browder's theorem first appeared in [**5**] implicitly, and the present form is found in [**1**].

Browder [**2**] has combined the two fixed point theorems and obtained a coincidence theorem [**2**, Theorem 3]. We point out that Browder's coincidence theorem is implicitly contained in Ha [**7**, Theorem 3] and obtain a generalization of the coincidence theorem with a simple proof using Ha's lemma.

THEOREM 1. *Let X be a nonempty convex subset of a linear topological space E and let Y be a nonempty compact convex subset of a linear topological space F. Let A be an upper semicontinuous and closed convex-valued mapping of X into Y.*

Received by the editors December 11, 1984.
1980 *Mathematics Subject Classification*. Primary 47H10, 49A40.
Key words and phrases. Coincidence theorem, saddle point, fixed point.

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0002-9939/86 $1.00 + $.25 per page

Let B be a convex-valued mapping of Y into X such that $B^{-1}(x)$ is open in Y for each x in X. Then there exist a point x_0 of X and a point y_0 of Y such that $y_0 \in A(x_0)$ and $x_0 \in B(y_0)$.

PROOF. By [**2**, Proposition 1] there exists a continuous mapping p of Y into the convex hull S of a finite number of points of X such that $p(y) \in B(y)$ for each y in Y. Then there exists a point x_0 of S such that $x_0 \in p(A(x_0))$ by [**7**, Lemma 2], and hence there exists a point y_0 of $A(x_0)$ such that $x_0 = p(y_0) \in B(y_0)$.

It is easily seen that Fan and Browder's theorem is derived from Theorem 1 by setting $X = Y$ with $E = F$ and A the identity mapping of X. Moreover we can easily derive Kakutani and Fan's theorem from Theorem 1:

Let $X = Y$ with $E = F$ and $B_U(y) = \{x \in X: (x, y) \in \Delta + U \times U\}$ for each y in X and each open convex zero-neighborhood U of E, where Δ is the diagonal $\{(x, x): x \in X\}$ of X. Then there exists a net $\{(x_U, y_U)\}$ directed by the system of open convex zero-neighborhoods of E such that (x_U, y_U) belongs to the set $\mathrm{Gr}(A) \cap \mathrm{Gr}(B_U)$ for each U. Since $\mathrm{Gr}(A)$ is compact, there exist a point (x_0, y_0) of $\mathrm{Gr}(A)$ and a subnet of $\{(x_U, y_U)\}$ converging to (x_0, y_0). Then x_0 must be equal to y_0 by the definition of B_U, and hence we have $x_0 \in A(x_0)$.

We can also derive from Theorem 1 the following theorem due to Simons [**10**] which has generalized and unified fixed point theorems for multi-valued mappings due to Browder [**2**], Fan [**6**], and Takahashi [**11, 12**].

THEOREM 2 [**10**, THEOREM 4.5]. *Let X be a nonempty compact convex subset of a linear topological space E and let B be a convex-valued mapping of X into the topological dual space E' of E such that $B^{-1}(x')$ is open in X for each x' in E'. Then there exist a point x_0 of X and a point x'_0 of E' such that $x'_0 \in B(x_0)$ and $\langle x_0, x'_0\rangle = \max_{x\in X}\langle x, x'_0\rangle$.*

PROOF. Let $A(x') = \{x \in X: \langle x, x'\rangle = \max_{x\in X}\langle x, x'\rangle\}$. We endow E' with the strong topology, that is, the uniform convergence topology on the bounded sets of E. It is easily seen that A is convex-valued. Let $\{(x_\alpha, x'_\alpha)\}$ be a net in $\mathrm{Gr}(A)$ converging to a point (x, x') of $X \times E'$. Since the net $\{x'_\alpha\}$ converges to x' uniformly on X, the net $\{\langle x_\alpha, x'_\alpha\rangle\}$ converges to $\langle x, x'\rangle$. Hence for any point u of X, we have the inequalities

$$\langle x, x'\rangle = \lim_\alpha \langle x_\alpha, x'_\alpha\rangle \geq \lim_\alpha \langle u, x'_\alpha\rangle = \langle u, x'\rangle.$$

Hence (x, x') belongs to $\mathrm{Gr}(A)$ and $\mathrm{Gr}(A)$ is closed. Therefore A is closed convex-valued and upper semicontinuous. Then we can apply Theorem 1 to the multi-valued mappings A and B and obtain the desired result.

We finish our discussion with a saddle point theorem which can be derived from Theorem 1. A real-valued function f on a convex set X is said to be *quasi-convex* if the set $\{x \in X: f(x) \leq r\}$ is convex for each real number r. If $-f$ is quasi-convex, then f is said to be *quasi-concave.* Let $\{f_\nu: \nu \in I\}$ be a family of real-valued functions on a topological space X. The family $\{f_\nu: \nu \in I\}$ is said to be *equicontinuous* if for any point x of X and any positive number δ, there exists a neighborhood U of x such that $|f_\nu(u) - f_\nu(x)| < \delta$ for all u in U and ν in I. We denote by $C(X)$ the Banach space of all real-valued bounded continuous functions with the uniform norm.

THEOREM 3. *Let X be a nonempty convex subset of a linear topological space and let Y be a nonempty compact convex subset of a linear topological space. Let f be a real-valued continuous function on the product space $X \times Y$ of X and Y which is quasi-concave in its first variable and quasi-convex in its second variable and satisfies $\sup_{x\in X}\min_{y\in Y} f(x,y) < +\infty$. Let the family $\{f(x,\cdot): x \in X\}$ of real-valued functions on Y be equicontinuous and closed in the Banach space $C(Y)$. Then f has a saddle point (x_0, y_0), that is, (x_0, y_0) satisfies the equations*

$$\max_{x\in X} f(x,y_0) = f(x_0,y_0) = \min_{y\in Y} f(x_0,y).$$

PROOF. By the equicontinuity of the family $\{f(x,\cdot): x \in X\}$, for any y in Y there exists an open neighborhood V_y of y in Y such that $|f(x,v) - f(x,y)| < 1/2$ for all v in V_y and all x in X. Then there exists a finite subset Z of Y such that $Y = \bigcup\{V_z: z \in Z\}$ by the compactness of Y. On the other hand, we have $\min_{y\in Y}\sup_{x\in X} f(x,y) < +\infty$ by [**7**, Theorem 4], and hence there exist a number M and y_0 in Y such that $f(x,y_0) < M$ for all x in X. Let $P_1 = \{z \in Z: y_0 \in V_z\}$ and

$$P_{i+1} = \left\{z \in Z: V_z \cap \bigcup\{V_w: w \in P_i\} \neq \varnothing\right\} \setminus \bigcup\{P_j: 1 \leq j \leq i\} \quad \text{for } i = 1,2,\ldots.$$

Then there exists a positive number n such that $P_1,\ldots,P_n$ are all nonempty and $P_{n+1}, P_{n+2},\ldots$ are all empty. We have $Z = P_1 \cup P_2 \cup \cdots \cup P_n$. In fact, if $Z' = Z \setminus P_1 \cup \cdots \cup P_n$ is not empty, then the two open sets $\bigcup\{V_z: z \in P_1 \cup \cdots \cup P_n\}$ and $\bigcup\{V_w: w \in Z'\}$ cover Y and they have no intersection, which contradicts the connectedness of Y. Hence for any y in Y there exists an integer m with $1 \leq m \leq n$ such that $y \in V_z$ for some z in P_m, and by the construction of $P_1,\ldots,P_n$, there exists a sequence $y_1,\ldots,y_m$ in Y with $y_m = y$ such that y_{i-1} and y_i belong to a neighborhood V_z for some z in P_i for $i = 1,\ldots,m$. Hence for any x in X we have

$$\begin{aligned} f(x,y) - f(x,y_0) &\leq (f(x,y_0) - f(x,y_1)) + \cdots + (f(x,y_{m-1}) - f(x,y_m)) \\ &\leq m \leq n. \end{aligned}$$

Therefore we have

$$f(x,y) \leq n + f(x,y_0) \leq n + M = M'$$

for all x in X and all y in Y.

If we set $g(y) = \sup_{x\in X} f(x,y)$, then g is a real-valued function on Y and continuous. If fact, let y_0 be a point of Y and $g(y_0) < r$. If we set $\delta = r - g(y_0) > 0$, then there exists neighborhood V of y_0 such that $f(x,v) - f(x,y_0) < \delta/2$ for all x in X and all v in V. By the definition of g, for any y in Y there exists x_y in X such that $f(x_y,y) > g(y) - \delta/2$. Hence for any v in V,

$$g(v) < f(x_v,v) + \delta/2 < f(x_v,y_0) + \delta \leq g(y_0) + \delta = r.$$

Hence g is upper semicontinuous. Since g is also lower semicontinuous, g is continuous.

We define a multi-valued mapping A of X into Y by

$$A(x) = \left\{y: f(x,y) = \min_{y\in Y} f(x,y)\right\}.$$

Then the graph $\mathrm{Gr}(A)$ of A is closed by the continuity of f and the upper semicontinuity of $\min_{y\in Y} f(\cdot,y)$. It is easily seen that A is convex-valued, and hence

A is closed convex-valued and upper semicontinuous. On the other hand, for any positive integer k we define a multi-valued mapping B_k of Y into X by

$$B_k(y) = \{x: f(x,y) > g(y) - 1/k\}.$$

Then B_k is convex-valued and $B_k^{-1}(x)$ is open in Y for all x in X. Therefore there exist x_k in X and y_k in Y such that $x_k \in B_k(y_k)$ and $y_k \in A(x_k)$. Then we have the inequalities

$$M' \geq f(x_k, y) \geq f(x_k, y_k) > g(y_k) - 1/k$$

for all y in Y. Since the sequence $\{g(y_k) - 1/k\}$ is bounded by the continuity of g, the sequence $\{f(x_k, \cdot)\}$ is bounded in the Banach space $C(Y)$. Hence from the Arzelà-Ascoli theorem we may assume that the sequence uniformly converges to a function of the type $f(x_0, \cdot)$ for some x_0 in X. Moreover we may assume that the sequence $\{y_k\}$ converges to a point y_0 of Y by the compactness of Y. Then we have $f(x_0, y_0) = \min_{y \in Y} f(x_0, y)$. In fact, if the equation does not hold, then there exists a number c such that $f(x_0, y_0) > c > \min_{y \in Y} f(x_0, y) = f(x_0, y_1)$ with some y_1 in Y. Since the sequence $\{f(x_k, y_1)\}$ converges to $f(x_0, y_1)$, we have

$$c > f(x_k, y_1) \geq \min_{y \in Y} f(x_k, y) = f(x_k, y_k)$$

for sufficiently large k. Since the sequence $\{f(x_k, y_k)\}$ converges to $f(x_0, y_0)$, we have $f(x_0, y_0) \leq c$, which is a contradiction. On the other hand, from the inequality $f(x_k, y_k) > g(y_k) - 1/k$, we have $f(x_0, y_0) \geq g(y_0)$, that is, $f(x_0, y_0) = \max_{x \in X} f(x, y_0)$.

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