PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 96, Number 4, April 1986

STABLE RANK OF THE DISC ALGEBRA

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ABSTRACT. We prove that the Bass stable rank of the disc algebra is one.

Let A be the disc algebra, consisting of functions analytic on the unit disc in $\mathbf{C}$ and continuous on its closure. We prove the following result:

THEOREM 1. *Suppose* $f_1, f_2 \in A$ *and* $|f_1(z)| + |f_2(z)| > 0$ *for all* z *with* $|z| \leq 1$. *Then there are* $g_1, g_2 \in A$ *with* $g_1^{-1} \in A$ *and* $g_1 f_1 + g_2 f_2 = 1$.

The question of whether Theorem 1 is true arose in recent work of Rieffel [**2**] on K-theory of C^*-algebras. The connection is that Theorem 1 amounts to a computation of the Bass stable rank of A. If R is any ring, then its Bass stable rank is by definition $\operatorname{bsr}(R) = \min(n$: whenever $r_1 \cdots r_{n+1} \in R$ and $\{r_j\}$ generate R as a left ideal, there are $b_1 \cdots b_n \in R$ such that $r_1 + b_1 r_{n+1} \cdots r_n + b_n r_{n+1}$ generate R as a left ideal). Since functions which generate A cannot all vanish at the same point, one obtains $\operatorname{bsr}(A) = 1$ by dividing the conclusion of Theorem 1 by g_1. Now in [**2**], Rieffel introduces another concept, the topological stable rank, $\operatorname{tsr}(R)$. It is defined when R is a Banach algebra and is $\operatorname{tsr}(R) = \min(n$: whenever $r_1 \cdots r_n \in R$ and $\delta > 0$ there are $b_1 \cdots b_n \in R$ such that $\{b_j\}$ generate R as a left ideal and $\|b_j - r_j\| < \delta)$. Rieffel leaves open whether $\operatorname{tsr}(R) = \operatorname{bsr}(R)$ for all Banach algebras R, but suggests that the disc algebra should provide a counterexample. Theorem 1 shows that it does since, as he points out, it is easily seen that $\operatorname{tsr}(A) = 2$.

Theorem 1 may be regarded as a variant of the Carleson corona theorem, which states that if $f_1 \cdots f_n$ are bounded analytic functions on $D = \{z \in \mathbf{C}: |z| < 1\}$ and $\Sigma |f_j(z)| > \delta > 0$ on D, then there are bounded analytic functions $g_1 \cdots g_n$ such that $\Sigma g_j f_j = 1$. Our proof of Theorem 1 is in some sense a modification of the proof of the corona theorem given in [**1**], although the assumption of continuity at the boundary eliminates the analytic difficulties in the argument. The corona theorem itself can be proved quickly by soft techniques when the functions $f_1 \cdots f_n$ are continuous up to the boundary (see [**3**, p. 396]), since one can identify the maximal ideal space of A with $\overline{D}$. However, this argument does not give $g_1^{-1} \in A$. We make use of the following

Claim. Suppose f_1 and f_2 are as in Theorem 1. Then there is $\delta > 0$ and a continuous function $F: \overline{D} \to \mathbf{C}$, Lipschitz on compact subsets of D, and such that

(1) $F(z) = f_1(z)$ if $|f_1(z)| < \delta$, $F(z) = 1$ if $|f_2(z)| < \delta$,

(2) $|F(z)| \geq \delta$ if $|f_1(z)| \geq \delta$,

(3) $\partial F / \partial \overline{z}$ is bounded on D.

To construct F choose closed sets E_1 and E_2 such that E_j has finitely many components and $\{|f_j| < \delta\} \subseteq E_j \subseteq \{|f_j| < 2\delta\}$, where δ is small enough so that

Received by the editors February 3, 1983 and, in revised form, January 8, 1985.
1980 *Mathematics Subject Classification*. Primary 46J15; Secondary 30D50, 46L99.
[1]The authors were partially supported by the NSF.

$\{|f_1| < 2\delta\} \cap \{|f_2| < 2\delta\} = \varnothing$. By the maximum principle, E_2 cannot separate any point of E_1 from the boundary of the disc. So we can extend the components of E_1 to the boundary to obtain a closed set $S \subseteq \overline{D}$ having the following properties: $S \supseteq E_1$, $S \cap E_2 = \varnothing$, and S has finitely many components each of which intersects ∂D. The components of $\overline{D} \backslash S$ are then simply connected, so a bounded continuous branch of $\log f_1$ exists on $\overline{D} \backslash S$. Since E_2 and S are compact and $E_2 \cap S = \varnothing$, there is a function $q \in C^\infty(R^2)$ such that $q = 1$ on a neighborhood of S and $q = 0$ on a neighborhood of E_2. Define $F(z) = \exp(q(z) \log f_1(z))$. Clearly F is continuous on $\overline{D}$ and satisfies (1) and (2). As for (3) we have $\partial F / \partial \overline{z} = F \log f_1 (\partial q / \partial \overline{z})$ when $z \notin S$ and zero otherwise, and q is smooth. The claim is proved.

To prove Theorem 1 let $g_1 = (F/f_1)\exp(uf_2)$, where u is as yet unknown. For g_1 to belong to A we need u continuous on $\overline{D}$ and satisfying

$$\frac{\partial u}{\partial \overline{z}} = \frac{1}{F f_2} \frac{\partial F}{\partial \overline{z}}$$

on D. Set

$$k = \frac{1}{F f_2} \frac{\partial F}{\partial \overline{z}}.$$

By (1) and (2), $|F f_2| \geq \delta^2$ if $\partial F / \partial \overline{z} \neq 0$. So k is bounded on D and since the convolution of a bounded function and an L^1 function is continuous,

$$u(z) = \frac{1}{\pi} \int \int_D \frac{k(\varsigma)}{\varsigma - z} \, d\varsigma \, d\overline{\varsigma}$$

has the desired properties.

Moreover, g_1 is bounded away from zero on D, so $g_1^{-1} \in A$. Let g_2 be determined by the condition $g_1 f_1 + g_2 f_2 = 1$. Clearly, g_2 is continuous when $f_2 \neq 0$. On the other hand, if $0 < |f_2| < \delta$, then

$$g_2 = \frac{1}{f_2}(1 - F \exp(uf_2)) = \frac{1}{f_2}(1 - \exp(uf_2))$$

by (1), so $g_2 \to -u$ as $f_2 \to 0$. This finishes the proof.

We are grateful to Donald Sarason for communicating the problem.

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