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TRANSFORMATIONS INDUCED IN THE STATE SPACE OF A C^*-ALGEBRA AND RELATED ERGODIC THEOREMS

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ABSTRACT. Let A be a norm-separable C^*-algebra with unit $\mathbf{1}$, σ-weakly dense in a W^*-algebra M, and let α be a positive linear mapping of M into itself leaving $\mathbf{1}$ invariant. We show that α induces a transformation $\tilde{\alpha}$ defined "almost everywhere" on the state space σ of A with values in σ. If α is a $*$-automorphism of M, then there exists

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\tilde{\alpha}_n\psi$$

for "almost all" states ψ of A, where $\tilde{\alpha}_n$ are transformations on σ induced by α^n.

1. Introduction. Let M be a W^*-algebra and let A be a norm-separable C^*-subalgebra of M with unit $\mathbf{1}$, σ-weakly dense in M. Let α be a positive linear mapping of M into itself such that $\alpha(\mathbf{1}) = \mathbf{1}$. The mapping α induces, in the natural way, a transformation α^*: $M^* \to M^*$, namely, $(\alpha^*\varphi)(x) = \varphi(\alpha x)$ for $\varphi \in M^*$, $x \in M$. However, for the purposes of quantum mechanics, it is desirable to define an induced transformation $\tilde{\alpha}$: $\sigma_0 \to \sigma$, where σ_0 is a (possibly large) subset of the state space σ of A. Let us discuss this question more thoroughly. In the algebraic formalism of quantum mechanics the observables of a physical system are represented by the selfadjoint elements of a (norm-separable) C^*-algebra A, and the states of the system by a subset of the state space σ of A. The dynamics of the system is usually assumed to be represented by a one-parameter group $\{\alpha'_t$: $t \in \mathbf{R}\}$ of $*$-automorphisms of A. However, in many important cases (e.g. for the noninteracting Bose gas), the assumption about the dynamics is not satisfied. Thus, as a more appropriate form of the dynamics, one considers a one-parameter group $\{\alpha_t$: $t \in \mathbf{R}\}$ of $*$-automorphisms of the von Neumann algebra $\pi_\omega(A)''$, where π_ω is the GNS representation of A associated with an equilibrium state ω of A (see [**1**, Introduction]). Assuming this form of the dynamics, let us identify A with $\pi_\omega(A)$ and put $M = \pi_\omega(A)''$. Then, if a state ψ on A is the restriction of a state φ of M, the expectation value $\psi(a)$ at instant zero would evolve in time t, in the Schrödinger picture, to $(\tilde{\alpha}_t\psi)(a) = (\alpha_t^*\varphi)(a)$ for any a in A. Now, the problem arises to define the evolution $\tilde{\alpha}_t\psi$ for sufficiently many ψ not of the above type, at least for a discrete time variable t, obtaining an orbit $\{\alpha_n\psi$: $n = 0, \pm 1,\dots\}$ which is the basic

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object of many ergodic problems. Having defined such an orbit, the ergodic problem in the formulation of [7] lies in what follows:

Prove, for as many states ψ of A as possible, the existence of

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \tilde{\alpha}_n \psi,$$

where the limit is taken in the weak*-topology of the state space of A.

Thus we are concerned with two questions: to construct a transformation $\tilde{\alpha}$ defined on some subset of the state space σ of A induced by a positive mapping α of M into itself (or, more generally, a sequence of transformations $\tilde{\alpha}_n$ induced by α^n) and to prove the ergodic theorem as formulated above. Both the problems were considered by Radin in [**6**, **7**], who solved them under the assumption that the von Neumann algebra M is finite. The aim of this paper is to generalize Radin's result to the case of an arbitrary algebra M.

2. Preliminaries and notation. Throughout the paper, M will denote a σ-finite W^*-algebra; A a norm-separable C^*-subalgebra of M with unit $\mathbf{1}$, σ-weakly dense in M; σ the state space of A; and ρ a normal faithful state of M. Considering the GNS representation of M associated with ρ, we assume that M and A are algebras of operators acting in a Hilbert space H.

Let $\{x_n\} \subset M$, $x \in M$. We say that x_n converges almost uniformly to x, $x_n \to x$ a.u., if, for each $\varepsilon > 0$, there is a projection e in M such that $\rho(e) \geqslant 1 - \varepsilon$ and $\lim_{n\to\infty}\|(x_n - x)e\| = 0$.

x_n converges quasi-uniformly to x, $x_n \to x$ q.u., if, for each $\varepsilon > 0$ and each projection e in M, there is a projection f in M such that $f \leqslant e$, $\rho(e - f) < \varepsilon$ and $\lim_{n\to\infty}\|(x_n - x)f\| = 0$.

It is obvious that q.u. convergence implies a.u. convergence and the result of Paszkiewicz [**4**] says that these two types of convergence are identical for bounded sequences of M.

A sequence $\{e_n\}$ of projections in M is called an exhaustion if $e_n \uparrow \mathbf{1}$ σ-weakly.

Let σ_0 be a subset of the state space σ of A. σ_0 is said to be of full measure if there is an exhaustion $\{e_n\}$ such that

$$\sigma_0 \supset \bigcup_{n=1}^{\infty} \left\{ \overline{\varphi|A \colon \varphi \in M_*^+, s(\varphi) \leqslant e_n} \right\},$$

where $s(\varphi)$ is the support of φ and the closure is taken with respect to the weak*-topology of σ (cf. [**7**]).

3. The noncommutative von Neumann-Maharam theorem. Let α be a positive linear mapping of M into itself such that $\alpha(\mathbf{1}) = \mathbf{1}$. A transformation $\tilde{\alpha}$ defined on a subset of the state space σ of A with values in σ will be called induced by α if

$$\tilde{\alpha}(\varphi|A) = (\alpha^*\varphi)\,|\,A$$

for all states φ of M such that $\varphi\,|\,A$ belongs to a weakly*-dense subset of the domain of $\tilde{\alpha}$. In this section we prove the noncommutative von Neumann-Maharam theorem which states that α induces a transformation $\tilde{\alpha}$ defined almost everywhere on σ (i.e. on a subset of σ of full measure).

For x in M, put $\|x\|_2 = [\rho(x^*x)]^{1/2}$.

LEMMA 1. *Let x_n, x belong to M, and assume that $\sum_{n=1}^{\infty}\|x_n - x\|_2^2 < \infty$. Then $x_n \to x$ a.u.*

PROOF. Denoting $y_n = (x_n - x)^*(x_n - x)$, we obtain a sequence $\{y_n\}$ of positive operators in M with $\sum_{n=1}^{\infty}\rho(y_n) < \infty$. Take a sequence $\{\varepsilon_n\}$ of positive numbers, $\varepsilon_n \downarrow 0$ and $\sum_{n=1}^{\infty}\varepsilon_n^{-1}\rho(y_n) < \infty$. Given $\varepsilon > 0$, we choose a positive integer N such that $\sum_{n=N}^{\infty}\varepsilon_n^{-1}\rho(y_n) < \varepsilon/2$ and consider the sequence $\{y_n\}_{n=N}^{\infty}$. According to [**2**, Theorem 1.2, p. 256, with $m = 1$], there is a projection e in M with the properties

$$\|ey_ne\| < \varepsilon_n \quad \text{for } n \geqslant N \quad \text{and} \quad \rho(e) \geqslant 1 - 2\sum_{n=N}^{\infty}\varepsilon_n^{-1}\rho(y_n).$$

Since $\|ey_ne\| = \|e(x_n - x)^*(x_n - x)e\| = \|(x_n - x)e\|^2$, we get that $\lim_{n\to\infty}\|(x_n - x)e\| = 0$ and $\rho(e) \geqslant 1 - \varepsilon$, which completes the proof.

LEMMA 2. *For each x in M, there exists a sequence $\{a_n\} \subset A$ such that $a_n \to x$ q.u.*

PROOF. Let $x \in M$ be given, and consider the ball $\{y \in M\colon \|y\| \leqslant \|x\|\}$ with the strong topology. This topology is metrizable by the metric $d(y_1, y_2) = \|y_1 - y_2\|_2$ ([**8**, Proposition 5.3, p. 148] together with the fact that, for bounded sets, the σ-strong and the strong topologies coincide). By virtue of the Kaplansky density theorem, the ball $\{a \in A\colon \|a\| \leqslant \|x\|\}$ is strongly dense in $\{y \in M\colon \|y\| \leqslant \|x\|\}$ and the metrizability yields that there is a sequence $\{b_k\}$ in A such that $\|b_k\| \leqslant \|x\|$ and $\|b_k - x\|_2 \to 0$. Take a subsequence $\{b_{k_n}\}$ with the property $\sum_{n=1}^{\infty}\|b_{k_n} - x\|_2^2 < \infty$ and put $a_n = b_{k_n}$. On account of Lemma 1, we have that $a_n \to x$ a.u. and, since $\|a_n\| \leqslant \|x\|$, the result of Paszkiewicz, mentioned in §2, implies that $a_n \to x$ q.u.

Now, we prove a generalization of the noncommutative Egorov theorem (cf. [**8**, Theorem 4.13, p. 85]).

THEOREM 3. *For each sequence $\{x_n\} \subset M$, there exist a sequence $\{a_{nj}\} \subset A$ and an exhaustion $\{e_k\}$ in M, such that*

$$\|(x_n - a_{nj})e_k\| \to 0 \quad \text{as } j \to \infty, \text{ for } n, k = 1, 2, \ldots.$$

PROOF. According to Lemma 2, we can find a sequence $\{a_{nj}\} \subset A$ with $a_{nj} \to x_n$ q.u. as $j \to \infty$, for each n. Our first step consists in showing that, for each projection q in M and each $\varepsilon > 0$, there is a projection $p \leqslant q$ in M with $\rho(q - p) \leqslant \varepsilon$, such that $\|(x_n - a_{nj})p\| \to 0$ as $j \to \infty$, for each n.

To this end, take an arbitrary $\varepsilon > 0$ and choose a projection $p_1 \leqslant q$ such that

$$\|(x_1 - a_{1j})p_1\| \to 0 \quad \text{as } j \to \infty \quad \text{and} \quad \rho(q - p_1) < \varepsilon/2.$$

This is possible because $a_{1j} \to x_1$ q.u. Next, we choose a projection $p_2 \leqslant p_1$ such that

$$\|(x_2 - a_{2j})p_2\| \to 0 \quad \text{as } j \to \infty \quad \text{and} \quad \rho(p_1 - p_2) < \varepsilon/4,$$

and so on.

Proceeding that way, we obtain a sequence $\{p_n\}$ of projections in M with $p_n \leqslant p_{n-1}$ and $\rho(p_{n-1} - p_n) < \varepsilon/2^n$.

Put $p = \lim_{n\to\infty} p_n$. p is a projection in M and, since

$$\rho(p_n) > \rho(p_{n-1}) - \varepsilon/2^n > \cdots > \rho(q) - (\varepsilon/2 + \cdots + \varepsilon/2^n) > \rho(q) - \varepsilon,$$

we have $\rho(p) \geqslant \rho(q) - \varepsilon$. Evidently, $p \leqslant q$ and, for each n,

$$\|(x_n - a_{nj})p\| \leqslant \|(x_n - a_{nj})p_n\| \to 0 \quad \text{as } j \to \infty,$$

which states our claim.

Now, let $\varepsilon_n \downarrow 0$ and let us find, according to the above considerations (with $q = \mathbf{1}$), a projection f_1 in M such that $\|(x_n - a_{nj})f_1\| \to 0$ as $j \to \infty$, for each n, and $\rho(\mathbf{1} - f_1) \leqslant \varepsilon_1$.

Putting $q = \mathbf{1} - f_1$ in the first part of the proof, we can find a projection $f_2 \leqslant \mathbf{1} - f_1$ such that

$$\|(x_n - a_{nj})f_2\| \to 0 \quad \text{as } j \to \infty,$$

for each n, and $\rho(\mathbf{1} - (f_1 + f_2)) \leqslant \varepsilon_2$.

Having found the projections $f_1, \ldots, f_{k-1}$, we can find a projection $f_k \leqslant \mathbf{1} - (f_1 + \cdots + f_{k-1})$ such that

$$\|(x_n - a_{nj})f_k\| \to 0 \quad \text{as } j \to \infty,$$

for each n, and $\rho(\mathbf{1} - (f_1 + \cdots + f_k)) \leqslant \varepsilon_k$. Thus, we obtain a sequence $\{f_k\}$ of projections in M, pairwise orthogonal and such that $\|(x_n - a_{nj})f_k\| \to 0$ as $j \to \infty$, for each n, k, and $\rho(f_1 + \cdots + f_k) \geqslant 1 - \varepsilon_k$.

To construct the desired exhaustion, put $e_k = f_1 + \cdots + f_k$. We have $e_k \leqslant e_{k+1}$ and $\rho(e_k) \geqslant 1 - \varepsilon_k \to 1$ as $k \to \infty$, which shows that $\{e_k\}$ is an exhaustion. Moreover, for each n, k,

$$\|(x_n - a_{nj})e_k\| \leqslant \|(x_n - a_{nj})f_1\| + \cdots + \|(x_n - a_{nj})f_k\| \to 0 \quad \text{as } j \to \infty,$$

which completes the proof.

Now, we are in a position to prove the main result of this section.

THEOREM 4. *Let α be a positive linear mapping of M into itself such that $\alpha(\mathbf{1}) = \mathbf{1}$. Then there exist a subset σ_0 of full measure in the state space σ and a transformation $\tilde{\alpha}$: $\sigma_0 \to \sigma$ induced by α.*

PROOF. Let $A_0 = \{a_1, a_2, \ldots\}$ be a countable norm-dense subset of A. Put $x_n = \alpha(a_n)$. By virtue of Theorem 3, there exist a sequence $\{a_{nj}\} \subset A$ and an exhaustion $\{e_k\}$ in M, such that $\|(x_n - a_{nj})e_k\| \to 0$ as $j \to \infty$ for each n, k.

Let us define σ_0 by the equality

$$\sigma_0 = \bigcup_{k=1}^{\infty} \left\{ \overline{\varphi|A \colon \varphi \in M_*^+, \|\varphi\| = 1, s(\varphi) \leqslant e_k} \right\},$$

where $s(\varphi)$ is the support of φ and the closure is taken with respect to the weak*-topology of $\sigma \supset \sigma_0$.

Let $\psi \in \sigma_0$. Then $\psi(a) = \lim_{\lambda \in \Lambda} \psi_\lambda(a)$ for all $a \in A$, and $\psi_\lambda = \varphi_\lambda | A$, where $\varphi_\lambda \in M_*^+$, $\|\varphi_\lambda\| = 1$ and $s(\varphi_\lambda) \leqslant e_K$ for some fixed K and all $\lambda \in \Lambda$.

The representation of ψ_λ as $\varphi_\lambda | A$ is unique since the equality $\varphi_\lambda^{(1)} | A = \varphi_\lambda^{(2)} | A$ yields $\varphi_\lambda^{(1)} = \varphi_\lambda^{(2)}$ because $\varphi_\lambda^{(1)}$ and $\varphi_\lambda^{(2)}$ are normal and A is σ-weakly dense in M. We are going to define $\tilde{\alpha}$ on σ_0 by

$$(\tilde{\alpha}\psi)(a) = \lim_{\lambda \in \Lambda} \varphi_\lambda(\alpha(a)).$$

First, we shall show that $\tilde{\alpha}\psi$ is well defined for all $a_n \in A_0$.

Given $\varepsilon > 0$, with fixed n and K as above, we find a number j_0 such that $\|(x_n - a_{nj_0})e_K\| < \varepsilon/3$. Since $a_{nj_0} \in A$, we have $\psi(a_{nj_0}) = \lim_{\lambda \in \Lambda} \psi_\lambda(a_{nj_0})$, which implies that

$$\left|\varphi_{\lambda'}(a_{nj_0}) - \varphi_{\lambda''}(a_{nj_0})\right| < \varepsilon/3$$

for sufficiently large λ', λ''. For these λ', λ'', we have

$$\begin{aligned}\left|\varphi_{\lambda'}(\alpha(a_n)) - \varphi_{\lambda''}(\alpha(a_n))\right| &= \left|\varphi_{\lambda'}(x_n) - \varphi_{\lambda''}(x_n)\right| \\ &\leqslant \left|\varphi_{\lambda'}(x_n) - \varphi_{\lambda'}(a_{nj_0})\right| + \left|\varphi_{\lambda'}(a_{nj_0}) - \varphi_{\lambda''}(a_{nj_0})\right| \\ &\quad + \left|\varphi_{\lambda''}(a_{nj_0}) - \varphi_{\lambda''}(x_n)\right| \\ &\leqslant 2\left\|(x_n - a_{nj_0})e_K\right\| + \left|\varphi_{\lambda'}(a_{nj_0}) - \varphi_{\lambda''}(a_{nj_0})\right|\end{aligned}$$

since $s(\varphi_{\lambda'})$, $s(\varphi_{\lambda''}) \leqslant e_K$. The above estimation yields the inequality $|\varphi_{\lambda'}(\alpha(a_n)) - \varphi_{\lambda''}(\alpha(a_n))| < \varepsilon$ proving the existence of $\lim_{\lambda \in \Lambda} \varphi_\lambda(\alpha(a_n))$.

Now let a_n, a_m be arbitrary elements of A_0. We have

$$\begin{aligned}\left|(\tilde{\alpha}\psi)(a_n) - (\tilde{\alpha}\psi)(a_m)\right| &= \lim_{\lambda \in \Lambda} \left|\varphi_\lambda(\alpha(a_n)) - \varphi_\lambda(\alpha(a_m))\right| \\ &\leqslant \lim_{\lambda \in \Lambda} \|\varphi_\lambda\| \|\alpha(a_n) - \alpha(a_m)\| \leqslant \|a_n - a_m\|.\end{aligned}$$

The last inequality shows that $\tilde{\alpha}\psi$ is uniformly continuous on A_0 and thus has the unique continuous extension to A. It is standard to show that this extension can be defined by the limiting procedure.

The fact that $\tilde{\alpha}\psi$ is a state on A is almost trivial. That $\tilde{\alpha}$ is induced by α is a consequence of the definition of $\tilde{\alpha}\psi$ as $\lim_{\lambda \in \Lambda}(\alpha^*\varphi_\lambda) | A$ and the fact that the set

$$\bigcup_{k=1}^{\infty} \left\{\varphi \middle| A \colon \varphi \in M_*^+, \|\varphi\| = 1, s(\varphi) \leqslant e_k\right\}$$

is weakly*-dense in σ_0.

Now, let $\mathbf{Z}$ be the set of all integers and $\mathbf{N}$ the set of nonnegative integers. Consider the semigroup $\{\alpha^n \colon n \in \mathbf{N}\}$ or the group $\{\alpha^n \colon n \in \mathbf{Z}\}$ for α invertible. We have

THEOREM 5. *There exists a family $\{\tilde{\alpha}_n \colon n \in \mathbf{N}$ (or $\mathbf{Z})\}$ of transformations defined on a subset of full measure $\sigma_0 \subset \sigma$ with values in σ, induced by $\{\alpha^n \colon n \in N$ (or $\mathbf{Z})\}$ in the sense that*

$$\tilde{\alpha}_n(\varphi | A) = ((\alpha^n)^*\varphi) | A$$

for $\varphi | A$ in a weakly-dense subset of σ_0.*

The proof follows the lines of the proof of Theorem 4; the only differences lie in choosing the exhaustion $\{e_k\}$ for the sequence $x_{nm} = \alpha^n(a_m)$ and defining $\tilde{\alpha}_n\psi$ as

$$(\tilde{\alpha}_n\psi)(a) = \lim_{\lambda\in\Lambda} \varphi_\lambda(\alpha^n(a)) \quad \text{for } \psi \in \sigma_0.$$

4. Ergodic theorems. In this section we assume that α is a *-automorphism of M leaving ρ invariant. The starting point in our considerations is the following refinement of Lance's ergodic theorem **[3]**, due to Petz **[5]**.

THEOREM 6. *There exists a norm-continuous linear projection* $\hat{}: M \to M$ *such that, for each* x *in* M, *the ergodic means*

$$S_N(x) = \frac{1}{N}\sum_{k=0}^{N-1} \alpha^k(x)$$

converge to $\hat{x}$ *q.u.*

Let us observe that, apart from its original proof, this result is an immediate consequence of Lance's theorem and the above-mentioned result of Paszkiewicz since $\|S_N(x)\| \leqslant \|x\|$.

PROPOSITION 7. *For each* $\varepsilon > 0$ *and each projection* q *in* M, *there exists a projection* $p \leqslant q$ *in* M *such that for each* $a \in A$, $\|(S_N(a) - \hat{a})p\| \to 0$ *and* $\rho(q - p) \leqslant \varepsilon$.

PROOF. Let $\varepsilon > 0$, let a projection q be given, and let $A_0 = \{a_1, a_2, \dots\}$ be a countable norm-dense subset of A. According to Theorem 6, we can find a projection $p_1 \leqslant q$ such that

$$\|(S_N(a_1) - \hat{a}_1)p_1\| \to 0 \quad \text{and} \quad \rho(q - p_1) < \varepsilon/2.$$

Again using Theorem 6, we find a projection $p_2 \leqslant p_1$ such that

$$\|(S_N(a_2) - \hat{a}_2)p_2\| \to 0 \quad \text{and} \quad \rho(p_1 - p_2) < \varepsilon/4.$$

Proceeding further in this way, we find a sequence $\{p_n\}$ of projections in M with the properties: $p_n \leqslant p_{n-1}$, $\|(S_N(a_n) - \hat{a}_n)p_n\| \to 0$ as $N \to \infty$ and $\rho(p_{n-1} - p_n) < \varepsilon/2^n$.

Put $p = \lim_{n\to\infty} p_n$. We have $p \leqslant q$ and since

$$\rho(p_n) > \rho(p_{n-1}) - \varepsilon/2^n > \cdots > \rho(q) - (\varepsilon/2 + \cdots + \varepsilon/2^n),$$

we obtain that $\rho(p) \geqslant \rho(q) - \varepsilon$. Moreover, for each $a_n \in A_0$,

$$\|(S_N(a_n) - \hat{a}_n)p\| \leqslant \|(S_N(a_n) - \hat{a}_n)p_n\| \to 0 \quad \text{as } N \to \infty$$

and the norm-density of A_0 in A together with the norm-continuity of $\hat{}$ yield the claim.

The following theorem answers one of the questions raised in **[7]** in a rather general form.

THEOREM 8. *There exists an exhaustion* $\{e_n\}$ *in* M *such that, for each* a *in* A,

$$\|(S_N(a) - \hat{a})e_n\| \to 0 \quad \text{as } N \to \infty \text{ for } n = 1, 2, \dots.$$

PROOF. Take a sequence of positive numbers $\{\varepsilon_n\}$, $\varepsilon_n \downarrow 0$. By virtue of Proposition 7, there is a projection p_1 in M such that, for each a in A,

$$\|(S_N(a) - \hat{a})p_1\| \to 0 \quad \text{and} \quad \rho(\mathbf{1} - p_1) \leqslant \varepsilon_1.$$

Again using Proposition 7, we find that there is a projection $p_2 \leqslant \mathbf{1} - p_1$ such that, for each a in A,

$$\|(S_N(a) - \hat{a})p_2\| \to 0 \quad \text{and} \quad \rho(\mathbf{1} - (p_1 + p_2)) \leqslant \varepsilon_2.$$

Proceeding further that way, we obtain a sequence $\{p_n\}$ of projections in M with the properties: $p_n \leqslant \mathbf{1} - (p_1 + \cdots + p_{n-1})$, $\|(S_N(a) - \hat{a})p_n\| \to 0$ as $N \to \infty$ for each a in A, and $\rho(\mathbf{1} - (p_1 + \cdots + p_n)) \leqslant \varepsilon_n$. Put $e_n = p_1 + \cdots + p_n$. We have $e_n \leqslant e_{n+1}$ and $\rho(e_n) \geqslant 1 - \varepsilon_n \to 1$ as $n \to \infty$, which shows that $\{e_n\}$ is an exhaustion. Moreover, for each a in A and every n,

$$\|(S_N(a) - \hat{a})e_n\| \leqslant \|(S_N(a) - \hat{a})p_1\| + \cdots + \|(S_N(a) - \hat{a})p_n\| \to 0$$

as $N \to \infty$, which completes the proof.

Let f be an arbitrary projection in M. A sequence $\{e_n\}$ of projections in M is called a conditional exhaustion with respect to f if $e_n \uparrow f$.

Following the lines of the proof of the above theorem, we could prove

PROPOSITION 9. *For each projection f in M, there exists a conditional exhaustion $\{e_n\}$ such that, for each a in A,*

$$\|(S_N(a) - \hat{a})e_n\| \to 0 \quad \text{as } N \to \infty \text{ for } n = 1, 2, \ldots.$$

As was stated before, there exist a subset σ_0 of σ,

$$\sigma_0 = \bigcup_{n=1}^{\infty} \left\{ \overline{\varphi|A \colon \varphi \in M_*^+, s(\varphi) \leqslant e_n} \right\}$$

for some exhaustion $\{e_n\}$ in M, and a family of transformations $\{\tilde{\alpha}_n\colon n = 0, \pm 1, \ldots\}$ defined on σ_0, induced by α. Our first goal is to reformulate Theorem 8 in the following way:

PROPOSITION 10. *There exists an exhaustion $\{f_n\}$ in M such that $f_n \leqslant e_n$ and, for each a in A,*

$$\|(S_N(a) - \hat{a})f_n\| \to 0 \quad \text{as } N \to \infty \text{ for } n = 1, 2, \ldots.$$

PROOF. Put $p_n = e_n - e_{n-1}$ ($e_0 = 0$). Then $\sum_{n=1}^{\infty} p_n = \mathbf{1}$, $p_n p_m = 0$ for $n \neq m$ and $e_n = \sum_{k=1}^{n} p_k$. For each p_n, we find, on account of Proposition 9, a conditional exhaustion $\{q_{mn}\}_{m=1}^{\infty}$ such that $q_{mn} \uparrow p_n$ as $m \to \infty$ and, for each a in A,

$$\|(S_N(a) - \hat{a})q_{mn}\| \to 0 \quad \text{as } N \to \infty,\ m = 1, 2, \ldots.$$

Let $f_m = \sum_{n=1}^{m} q_{mn}$. f_m is a projection as a sum of orthogonal projections. Moreover

$$f_m = \sum_{n=1}^{m} q_{mn} \leqslant \sum_{n=1}^{m} q_{m+1\,n} \leqslant \sum_{n=1}^{m+1} q_{m+1\,n} = f_{m+1}$$

and

$$f_m = \sum_{n=1}^{m} q_{mn} \leqslant \sum_{n=1}^{m} p_n = e_m.$$

To prove that $f_m \uparrow \mathbf{1}$, take an arbitrary $\varepsilon > 0$. We shall show that there exists a number m' such that $\rho(f_{m'}) \geqslant 1 - 2\varepsilon$ which is enough since the sequence $\{f_m\}$ is nondecreasing.

Choose a number m_0 such that $\sum_{k=1}^{m_0} \rho(p_k) \geqslant 1 - \varepsilon$. Next, choose numbers m_k, $k = 1, \dots, m_0$, such that

$$\rho(q_{m_k k}) \geqslant \rho(p_k) - \varepsilon/2^k \quad \text{for } k = 1, \dots, m_0,$$

which is possible because $q_{mn} \uparrow p_n$. Put $m' = \max(m_0, \dots, m_{m_0})$. We have

$$\begin{aligned} \rho(f_{m'}) &\geqslant \rho(q_{m_1 1}) + \cdots + \rho(q_{m_{m_0} m_0}) \\ &\geqslant \rho(p_1) + \cdots + \rho(p_{m_0}) - (\varepsilon/2 + \cdots + \varepsilon/2^{m_0}) \geqslant 1 - 2\varepsilon, \end{aligned}$$

the first inequality being a consequence of

$$q_{m''n} \geqslant q_{m'''n} \quad \text{for } m'' \geqslant m''',\ n = 1, 2, \dots.$$

The proof of the proposition has thus been completed.

Our last theorem provides a solution to the main problem raised in [7].

THEOREM 11. *For almost every state ψ in σ, the following limit, in the weak*-topology of σ, exists*:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \tilde{\alpha}_n \psi.$$

PROOF. Let σ_0 with the exhaustion $\{e_n\}$ be as defined before Proposition 10 and let $\{f_n\}$ be the exhaustion whose existence was proved in Proposition 10. Put

$$\sigma_0' = \bigcup_{n=1}^{\infty} \left\{ \overline{\varphi | A \colon \varphi \in M_*^+, \|\varphi\| = 1, s(\varphi) \leqslant f_n} \right\},$$

where the closure is taken with respect to the weak*-topology of σ. σ_0' is of full measure and, for an arbitrary element ψ of σ_0', we have $\psi(a) = \lim_\lambda \varphi_\lambda(a)$ for each a in A, where $\varphi_\lambda \in M_*^+$, $\|\varphi_\lambda\| = 1$ and $s(\varphi_\lambda) \leqslant f_{n_0}$ for a fixed n_0 and all λ.

For each a in A and positive integers N_1, N_2, we have, on account of Theorem 5,

$$\begin{aligned} &\left| \frac{1}{N_1} \sum_{n=0}^{N_1 - 1} (\tilde{\alpha}_n \psi)(a) - \frac{1}{N_2} \sum_{n=0}^{N_2 - 1} (\tilde{\alpha}_n \psi)(a) \right| \\ &\quad = \lim_\lambda \left| \frac{1}{N_1} \sum_{n=0}^{N_1 - 1} \varphi_\lambda(\alpha^n(a)) - \frac{1}{N_2} \sum_{n=0}^{N_2 - 1} \varphi_\lambda(\alpha^n(a)) \right| \\ &\quad = \lim_\lambda \left| \varphi_\lambda\big((S_{N_1}(a) - \hat{a}) f_{n_0} - (S_{N_2}(a) - \hat{a}) f_{n_0} \big) \right| \\ &\quad \leqslant \left\| (S_{N_1}(a) - \hat{a}) f_{n_0} - (S_{N_2}(a) - \hat{a}) f_{n_0} \right\|. \end{aligned}$$

By virtue of Proposition 10, the last expression tends to zero as N_1, $N_2 \to \infty$, and our result follows from the completeness of σ in the weak*-topology.

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