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FREQUENCY ENTRAINMENT FOR ALMOST PERIODIC EVOLUTION EQUATIONS

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ABSTRACT. A theorem of Massera stating that periodic solutions of equations are (under a simple hypothesis) entrained is generalized to limit periodic equations and (with a weak definition of entrainment) to almost periodic equations. An error is explained in a stronger result claimed by Cartwright.

Forbat [**3**, p. 166n; **4**, p. 30] has given an example of a system of differential equations, periodic in time, having a solution which is not entrained to the "forcing" frequency. The system is

$$\dot{x} = y, \qquad \dot{y} = -x + \left(x^2 + y^2 - 1\right)\sin\sqrt{2}\,t,$$

which admits the solution $x = \sin t$, $y = \cos t$ whose frequency 1 is incommensurable with (not entrained to) the frequency $\sqrt{2}$ of the system itself. This solution passes through no point at which the "forcing" with frequency $\sqrt{2}$ is felt. A little-known theorem of Massera, contained in [**5**], states that this is the only way in which entrainment can fail. The purpose of this paper is to give a proof of Massera's theorem which generalizes to limit periodic systems (and, to a lesser extent, to almost periodic (a.p.) systems), to evolution equations in Banach spaces, and to delay differential equations. Cartwright [**2**] has claimed a theorem much stronger than ours, but her theorem is false, as we shall show. (This does not affect her much deeper theorem, which is the main result of [**2**], and which has now been proved by a variety of methods. This theorem is complementary to ours in a way which will be discussed.)

LEMMA. *Let B be a Banach space and let $f\colon \mathbf{R}^2 \to B$ be an a.p. function of two variables (in the sense of* [**1**]*). Let M and N be modules of real numbers over the rationals such that M contains the frequencies of $f(s,t)$ in s, and N contains the frequencies in t; that is, such that the Fourier series of f can be written*

$$f(s,t) \sim \sum_{\substack{\mu\in M\\ \nu\in N}} a_{\mu\nu} e^{i(\mu s + \nu t)}.$$

Furthermore, suppose that

(i) *There is at least one value of s for which $f(s,t)$ is not independent of t.*

(ii) *$f(t,t)$, which is automatically an a.p. function, has its frequencies contained in M.*

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Then the intersection of M and N is larger than $\{0\}$, and if, in addition,

(iii) *M and N each have a single generator over the rationals,*

then $M = N$.

(REMARK. Since $f(s, t)$ is a uniform limit of trigonometric polynomials in two variables by the Approximation Theorem in [**1**], it follows that $f(t, t)$ is a uniform limit of trigonometric polynomials in one variable and is therefore a.p. as stated in (ii).)

PROOF. Hypothesis (i) implies that there exists a pair (μ, ν) with $\nu \neq 0$ such that $a_{\mu\nu} \neq 0$. According to (ii) there is an expansion

$$f(t, t) \sim \sum_{\mu \in M} b_\mu e^{i\mu t}$$

so that

$$0 \sim \sum (a_{\mu 0} - b_\mu) e^{i\mu t} + \sum_{\nu \neq 0} a_{\mu\nu} e^{i(\mu+\nu)t}.$$

Suppose now that $M \cap N = \{0\}$. In this case each term in the last equation has a frequency different from every other term, so that the expression (as it stands, without combining any terms) is the Fourier series of an a.p. function. It follows from the uniqueness theorem for a.p. functions that $a_{\mu 0} = b_\mu$ and $a_{\mu\nu} = 0$ for $\nu \neq 0$. Since this contradicts the existence of $a_{\mu\nu} \neq 0$ with $\nu \neq 0$, we conclude that $M \cap N \neq \{0\}$. It follows immediately that $M = N$ in the case where (iii) holds. Q.E.D.

REMARK. The example (with $B = \mathbf{C}$, the complex numbers)

$$f(s, t) = \exp\{i[(1 + \sqrt{2})s + (2 + \sqrt{3})t]\} - \exp\{i[(2 + \sqrt{2})s + (1 + \sqrt{3})t]\}$$

shows that neither $M \subseteq N$ nor $N \subseteq M$ follows from (i) and (ii) alone. Here (ii) is satisfied since $f(t, t) = 0$; M has generators 1 and $\sqrt{2}$; N has generators 1 and $\sqrt{3}$.

THEOREM. *Let F: $B \times \mathbf{R} \to B$ be a continuous mapping such that $F(u, t)$ is a.p. in t uniformly with respect to u for $\|u\| \leqslant K$. Let N be the smallest module containing the frequencies of F. Let ϕ: $\mathbf{R} \to B$ be an almost periodic solution, with $\|\phi(t)\| \leqslant K$, of the differential equation $\dot{u} = F(u, t)$, and let M be the smallest module containing its frequencies. Then either*

(a) *$F(\phi(s), t)$ is independent of t for every s or*

(b) *$M \cap N \neq \{0\}$.*

Furthermore, if F and ϕ are periodic or (more generally) limit periodic, then either (a) *holds or $M = N$ (in which case we say the solution ϕ is entrained).*

PROOF. It follows from remarks in Cartwright [**2**, pp. 172–173] that the function $f(s, t) = F(\phi(s), t)$ is an a.p. function of two variables with frequencies contained in M (for s) and N (for t). Since ϕ is a solution of the differential equation, $\dot{\phi}(t) = f(t, t)$, which is a.p., as remarked after the lemma. If an a.p. function has an a.p. derivative, the module of the derivative is contained in the module of the function; therefore the module of $f(t, t)$ is contained in M. We have shown that f satisfies the general requirements of the lemma and also hypothesis (ii). Now if (a) of the theorem is denied, then (i) of the lemma is affirmed, and (b) follows. In the limit periodic case, (iii) of the lemma also holds and the conclusion is stronger. Q.E.D.

REMARK 1. The significance of $M \cap N \neq \{0\}$ is that at least one frequency present in the differential equations must also appear in the solution. In the periodic case $M = N$ means that the least periods have a rational ratio, not that they are equal.

REMARK 2. Under essentially the same hypothesis Cartwright [**2**] proves that if $B = \mathbf{R}^n$ then (in the setting of our theorem) M is contained in a module generated by N and at most $n - 1$ additional elements. This upper bound on M has been proved in several ways (for instance [**6**]) and is certainly true. However, she also concludes that the frequencies of F are integrally dependent on those of ϕ, implying $N \subseteq M$. Her error occurs in the second sentence on p. 174, where she does not take account of cancellations which may occur, as in our example following the proof of the lemma. But even apart from this her result is not plausible because F could have frequencies that do not appear along the orbit of ϕ, and no one would expect these to appear in ϕ. It remains an open question whether F can have frequencies on the orbit of ϕ which do not appear in ϕ, or, in other words, whether the cancellations occurring in our example can occur in a differential equations context. Such an example does not yet exist.

REMARK 3. It is essential to the argument that F be continuous and defined on all of B (or at least all of $\|u\| \leqslant K$). For instance, the argument is applicable to a heat equation

$$u_t = u_{xx} + g(u, x, t) \quad \text{for } 0 \leqslant x \leqslant 1$$

if formulated for u in C^2 or a suitable Sobolev space but not for u in a space in which $\partial^2/\partial x^2$ is densely defined and unbounded.

For finite-dimensional delay differential equations it is probably simpler to modify the argument so that it is directly applicable rather than to reformulate the equation as a differential equation in a Banach space. For example, suppose $\phi(t)$ is a periodic solution of $\dot{u} = F(u(t), u(t - \tau), t)$ with F periodic in t. Taking $f(s, t) = F(\phi(s), \phi(s - \tau), t)$, we see that the solution must be entrained unless this function is independent of t for all s; in particular, any periodic solutions which exist must be entrained if F_t is nowhere vanishing.

REFERENCES

1. A. S. Besicovitch, *Almost periodic functions*, Dover, New York, 1954.

2. M. L. Cartwright, *Almost periodic differential equations and almost periodic flows*, J. Differential Equations **5** (1969), 167–181.

3. N. Forbat, *Amalytische Mechanik der Schwingungen*, V.E.B. Deutsche Verlag der Wissenschaft, Berlin, 1966.

4. P. Hagedorn, *Non-linear oscillations*, translated by W. Stadler, Clarendon Press, Oxford, 1981.

5. J. Massera, *Observaciones sobre las soluciones periodicas de ecuaciones diferenciales*, Bol. Fac. Ingen. Montevideo **4** (1950–53), 37–45.

6. G. O'Brien, *The frequencies of almost periodic solutions of almost periodic differential equations*, J. Austral. Math. Soc. **17** (1974), 332–344.

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