

A GAUSSIAN MEASURE FOR CERTAIN CONTINUED FRACTIONS

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ABSTRACT. We solve a variant of Gauss' problem for grotesque continued fraction using the approach of dependence with complete connections.

1. Introduction. For a very readable account of Gauss' introduction of a measure to study ordinary continued fraction expansions, see [3]. In [7] Rieger introduced the grotesque continued fraction expansion for an irrational number y of the interval $Y = [G - 2, G]$, with $G = (\sqrt{5} + 1)/2$, as follows. Let $\varepsilon_1(y) = \text{sign } y$. Then $\varepsilon_1 y^{-1}$ can be written uniquely as $\varepsilon_1 y^{-1} = \alpha_1(y) + T(y)$ with $\alpha_1 \geq 1$, $\alpha_1 \equiv 1 \pmod{2}$. Because $T(y) \in Y$ the above process can be inductively continued and as a consequence we get

$$(1.1) \quad y = \frac{\varepsilon_1(y)}{\alpha_1(y) + T(y)} = \frac{\varepsilon_1(y)}{\alpha_1(y) + \frac{\varepsilon_2(y)}{\alpha_2(y) + \dots}}$$

$$= \left[\begin{array}{c} \varepsilon_1(y), \varepsilon_2(y), \dots \\ \alpha_1(y), \alpha_2(y), \dots \end{array} \right],$$

where $\alpha_n \in N$, $\alpha_n \equiv 1 \pmod{2}$, $\varepsilon_n \in \{-1, 1\}$, $\alpha_n + \varepsilon_n > 1$, $n = 1, 2, \dots$. The expression (1.1) is called the *grotesque continued fraction expansion* of y . The iterated maps of T are

$$T^n y = \left[\begin{array}{c} \varepsilon_{n+1}(y), \dots \\ \alpha_{n+1}(y), \dots \end{array} \right], \quad n = 1, 2, \dots,$$

with $T^0 y = y$.

Let μ be a nonatomic probability measure on the σ -algebra of the Borel sets of Y and define

$$V_n(w) = V_n(w, \mu) = \mu(T^n < w), \quad n = 1, 2, \dots,$$

with $V_0(w) = \mu([G - 2, w])$. Here, using the approach of dependence with complete

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connections, we find the limit

$$\lim_{n \rightarrow \infty} \mu(T^n < w) = 1$$

and estimate the error $(T^n < w) - 1$ to be $O(q^n)$. Our result generalizes and improves that which Rieger obtained using another method (see [8]) for $\mu = (1/2)\lambda$, where λ is the Lebesgue measure, with convergence rate of $O(q^{n/2})$. Throughout we need the notation

$$N^* = \{1, 2, 3, \dots\},$$

$$N = \{0, 1, 2, \dots\},$$

R = the set of real numbers,

$[a]$ = the integral part of $a \in R$,

$$G = (\sqrt{5} + 1)/2,$$

Y = the set of irrationals of $[G - 2, G]$,

\mathcal{B}_Y = the σ -algebra of the Borel sets of Y ,

$\mathcal{P}(X)$ = the power set of X ,

$L(Y)$ = the space of the real-valued and Lipschitz functions defined on Y .

2. The Gauss-type relation. In the following we require the definitions and terminology of [2, 4 and 5]. Define

$$X = \{(x, \varepsilon) : x \in N^*, x \equiv 1 \pmod{2}, |\varepsilon| = 1, x + \varepsilon > 1\}.$$

Then we have

PROPOSITION 2.1 (GAUSS-TYPE RELATION). *For each $n \in N$, V_n satisfies the relation*

$$V_{n+1}(w) = \sum_{(x,\varepsilon) \in X} \varepsilon \left(V_n \left(\frac{\varepsilon}{x + G - 2} \right) - V_n \left(\frac{\varepsilon}{x + w} \right) \right), \quad w \in [G - 2, G].$$

PROOF. We remark that

$$T^n = \frac{\varepsilon_{n+1}}{\alpha_{n+1} + T^{n+1}}.$$

Therefore

$$\begin{aligned} V_{n+1}(w) &= \mu \left(\frac{1}{\alpha_{n+1} + w} < T^n < \frac{1}{\alpha_{n+1} + G - 2} \right) \\ &\quad + \mu \left(\frac{-1}{\alpha_{n+1} + G - 2} < T^n < \frac{-1}{\alpha_{n+1} + w} \right) \end{aligned}$$

and the desired result follows.

We now assume that V'_0 exists and is bounded (μ has bounded density). By induction we have that $V'_n, n \in N^*$, exist and are bounded as well. By differentiating the above Gauss-type equation we obtain

$$(2.1) \quad V'_{n+1}(w) = \sum_{(x,\varepsilon) \in X} \frac{1}{(x + w)^2} V'_n \left(\frac{\varepsilon}{x + w} \right).$$

Let us introduce the functions f_n in (2.1) where

$$f_n(w) = \frac{V'_n(w)}{\rho(w)}, \quad n \in N,$$

with $\rho(w) = 1/(w + 1)$, $w \in [G - 2, G]$. Then (2.1) becomes

$$f_{n+1}(w) = \sum_{(x, \epsilon) \in X} \frac{w + 1}{(w + x)(w + x + \epsilon)} f_n\left(\frac{\epsilon}{x + w}\right).$$

PROPOSITION 2.2. *The function*

$$P(w, (x, \epsilon)) = \frac{w + 1}{(w + x)(w + x + \epsilon)}$$

defines a transition probability function from (Y, \mathcal{B}_Y) to $(X, \mathcal{P}(X))$.

PROOF. We have to verify that

$$(2.2) \quad \sum_{(x, \epsilon)} P(w, (x, \epsilon)) = 1$$

for all $w \in Y$. To this end we write the left side of (2.2) as

$$(w + 1) \sum_{x=1,3,5,\dots} \left[\left(\frac{1}{w + x} - \frac{1}{w + x + 1} \right) + \sum_{x=3,5,\dots} \left(\frac{1}{w + x - 1} - \frac{1}{w + x} \right) \right]$$

and the desired equality follows.

Proposition 2.2 allows us to consider the random system with complete connections (RSCC) (see [2])

$$(2.3) \quad \{(Y, \mathcal{Y}), (X, \mathcal{X}), u, P\}$$

where

$$\mathcal{Y} = \mathcal{B}_Y, \quad \mathcal{X} = \mathcal{P}(X), \quad u(w, (x, \epsilon)) = \frac{\epsilon}{x + w},$$

$$P(w, (x, \epsilon)) = \frac{w + 1}{(w + x)(w + x + \epsilon)}.$$

Further, we denote by U the associated Markov operator of the RSCC (2.3) with the transition probability function $Q(\cdot, \cdot)$. Then $Q^n(\cdot, \cdot)$ will be the transition probability function after n paths of the same Markov chain (see [4]).

3. The ergodicity of the random system. Let us consider the norm $\|\cdot\|_L$ defined on $L(Y)$ by

$$\|f\|_L = \sup_{y \in Y} |f(y)| + \sup_{y' \neq y''} \frac{|f(y') - f(y'')|}{|y' - y''|}, \quad f \in L(Y).$$

As is well known, $(L(Y), \|\cdot\|_L)$ is a Banach algebra. The following proposition can be proved in the same manner as in [5].

PROPOSITION 3.1. *The RSCC (2.3) is uniformly ergodic.*

Moreover, Theorem 2.1.57 of Iosifescu-Theodorescu [2] implies that $Q^n(\cdot, \cdot)$ converges uniformly to a probability Q^∞ and that there exist two positive constants $q < 1$ and c such that

$$(3.1) \quad \|U^n f - U^\infty f\|_L \leq cq^n$$

for all $n \in N^*$, $f \in L(Y)$, where

$$(3.2) \quad U^n f(\cdot) = \int_Y f(y) Q^n(\cdot, dy), \quad U^\infty f = \int_Y f(y) Q^\infty(dy).$$

By virtue of Lemma 2.1.58 of Iosifescu-Theodorescu [2], U has no eigenvalues of modulus 1 other than 1.

Let $B(Y, \mathcal{Y})$ be the Banach space of all real-valued bounded and \mathcal{Y} -measurable functions defined on Y with the uniform norm $|\psi| = \sup_{y \in Y} |\psi(y)|$. Then, taking into account Proposition 2.1.6 of [2], the adjoint U^* of the operator U (with the transition probability function $Q(\cdot, \cdot)$) in $B(Y, \mathcal{Y})$ is

$$(U^* \mu)(\cdot) = \int_Y Q(y, \cdot) \mu(dy),$$

where μ belongs to the Banach space of all bounded additive set functions on \mathcal{Y} with bounded variation with the norm given by the total variation.

By virtue of Proposition 3.1, Theorem 3.3.3 of [2] says that Q^∞ is the only eigenvector of U^* and as a consequence

$$(3.3) \quad \int_{G-2}^G Q(y, B) Q^\infty(dy) = Q^\infty(B)$$

for all the Borel sets of Y .

Further, we determine the form of Q^∞ .

PROPOSITION 3.2. *The probability Q^∞ has the density*

$$(3.4) \quad \rho(w) = \frac{1}{w+1}, \quad w \in [G-2, G]$$

with the normalizing factor $1/(3 \log G)$.

PROOF. We have

$$Q(y, B) = \sum_{\substack{(x, \varepsilon) \in X \\ \varepsilon(x+y)^{-1} \in B}} P(y, (x, \varepsilon)),$$

for all $y \in Y$ and $B \in \mathcal{B}_Y$. By virtue of uniqueness of Q^∞ we have to show that the measure with density ρ defined in (3.4) satisfies the equality (3.3). Because the intervals $[u, G]$, $0 \leq u < G$, and $[G-2, v]$, $G-2 < v \leq 0$, generate \mathcal{B}_Y , it suffices to verify the equality (3.3) only for $B = [u, G]$ and $B = [G-2, v]$.

Let us consider $B = [u, G], 0 \leq u < G$. Suppose first that $[u^{-1}]$ is an even number. Then

$$\begin{aligned} \int_{G-2}^G Q(y, [u, G])\rho(y) dy &= \int_{G-2}^G \left(\sum_{\substack{x \equiv 1 \pmod{2} \\ u \leq (x+y)^{-1} \leq G}} P(y, (x, 1)) \right) \rho(y) dy \\ &= \int_{G-2}^G \left(\sum_{\substack{x \equiv 1 \pmod{2} \\ 1 \leq x \leq [u^{-1} - G]}} P(y, (x, 1)) \right) \rho(y) dy \\ &= \int_{u^{-1} - [u^{-1}] - 1}^G \left(\sum_{\substack{x=1 \\ x \equiv 1 \pmod{2}}}^{[u^{-1}] - 1} P(y, (x, 1)) \right) \rho(y) dy \\ &\quad + \int_{G-2}^{u^{-1} - [u^{-1}] - 1} \left(\sum_{\substack{x=1 \\ x \equiv 1 \pmod{2}}}^{[u^{-1}] + 1} P(y, (x, 1)) \right) \rho(y) dy \\ &= \int_{u^{-1} - [u^{-1}] - 1}^G \left(\frac{1}{y+1} - \frac{1}{y+2} + \dots + \frac{1}{y+[u^{-1}] - 1} - \frac{1}{y-[u^{-1}]} \right) dy \\ &\quad + \int_{G-2}^{u^{-1} - [u^{-1}] - 1} \left(\frac{1}{y+1} - \frac{1}{y+2} + \dots + \frac{1}{y+[u^{-1}] + 1} - \frac{1}{y+[u^{-1}] + 2} \right) dy \\ &= \log \frac{Gu^{-1}}{(G-1)(u^{-1} + 1)} = \log \frac{G+1}{u+1} = \int_u^G \rho(y) dy. \end{aligned}$$

If $[u^{-1}]$ is odd we obtain

$$\begin{aligned} \int_{G-2}^G Q(y, [u, G])\rho(y) dy &= \int_{G-2}^{u^{-1} - [u^{-1}]} \left(\sum_{\substack{x=1 \\ x \equiv 1 \pmod{2}}}^{[u^{-1}]} P(y, (x, 1)) \right) \rho(y) dy \\ &\quad + \int_{u^{-1} - [u^{-1}]}^G \left(\sum_{x=1}^{[u^{-1}] - 2} P(y, (x, 1)) \right) \rho(y) dy \\ &= \int_{G-2}^{u^{-1} - [u^{-1}]} \left(\frac{1}{y+1} - \frac{1}{y+2} + \dots + \frac{1}{y+[u^{-1}]} - \frac{1}{y+[u^{-1}] + 1} \right) dy \\ &\quad + \int_{u^{-1} - [u^{-1}]}^G \left(\frac{1}{y+1} - \frac{1}{y+2} + \dots + \frac{1}{y+[u^{-1}] - 2} - \frac{1}{y+[u^{-1}] - 1} \right) dy \\ &= \log \frac{G+1}{u+1} = \int_u^G \rho(y) dy. \end{aligned}$$

Finally, the proof for $B = [G - 2, v], G - 2 < v \leq 0$, is similar.

4. The Gauss-type theorem. Now we are prepared to find the limit of $\mu(T^n < w)$ as $n \rightarrow \infty$ and the rate of convergence.

PROPOSITION 4.1. *If μ has a Riemann integrable density, then*

$$\lim_{n \rightarrow \infty} \mu(T^n < w) = \frac{1}{3 \log G} \log(G(w + 1)), \quad w \in [G - 2, G].$$

If the density of μ is a Lipschitz function, then there exist two positive constants c and $q < 1$ such that for all $w \in [G - 2, G]$, $n \in N^$,*

$$\mu(T^n < w) = \frac{1}{3 \log G} (1 + \theta q^n) \log(G(w + 1)),$$

where $\theta = \theta(\mu, n, w)$ with $|\theta| \leq c$.

PROOF. If V'_0 is a Lipschitz function, then $f_0 \in L(Y)$. By virtue of (3.2) we have

$$U^\infty f_0 = \int_{G-2}^G f_0(y) Q^\infty(dy) = \frac{1}{3 \log G}.$$

Taking into account (3.1) there exist two constants c and $q < 1$ such that

$$\|U^n f_0 - U^\infty f_0\|_L \leq cq^n \quad \text{for all } n \in N^*.$$

Further, consider $C(Y)$ to be the metric space of real continuous functions defined on Y with the norm $\|\cdot\| = \sup|\cdot|$. As $L(Y)$ is a dense subset of $C(Y)$ we have

$$(4.1) \quad \lim_{n \rightarrow \infty} \|(U^n - U^\infty)f_0\| = 0$$

for all $f_0 \in C(Y)$. Therefore (4.1) remains valid for measurable f_0 that are Q^∞ -almost surely continuous, that is, for Riemann-integrable f_0 .

We have

$$\begin{aligned} \lim_{n \rightarrow \infty} \mu(T^n < w) &= \lim_{n \rightarrow \infty} \int_{G-2}^w U^n f_0(u) \cdot \rho(u) du = \frac{1}{3 \log G} \int_{G-2}^w \rho(u) du \\ &= \frac{1}{3 \log G} \log \frac{w + 1}{G - 1} = \frac{1}{3 \log G} \log(G(w + 1)). \end{aligned}$$

COROLLARY 4.2. *Let $m \in N^*$, $(i_1, j_1), \dots, (i_m, j_m) \in X$ be fixed and take*

$$\mu(\cdot) = \frac{1}{2} \lambda(\cdot | \alpha_r = i_r, \varepsilon_r = j_r, r = 1, \dots, m).$$

Then

$$\lambda(T^{n+m} < w | a_r = i_r, \varepsilon_r = j_r, r = 1, \dots, m) = (1 + \theta_m q^n) \frac{2}{3 \log G} \log(G(w + 1))$$

where $\theta_m = \theta_m(n, w, s_m)$ with $s_r = j_r(i_r + s_{r-1})^{-1}$, $r = 1, \dots, m$, $s_0 = 0$.

REMARK. The study of optimality of the rate of convergence in Proposition 4.1 remains an open question. Concerning this, Wirsing's method for the classical Gauss' problem cannot be applied in our case because the corresponding operator is not a positive one, see [10].

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