PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 96, Number 4, April 1986

A_p-WEIGHT PROPERTIES OF REAL ANALYTIC FUNCTIONS IN $\mathbf{R}^n$

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ABSTRACT. In this note we show that given any real analytic function u in $\mathbf{R}^n$, there exists some $p > 1$ for which $|u|$ is locally an A_p-weight of Muckenhoupt.

1. Statement of the results. Let w be a nonnegative $L^1_{\mathrm{loc}}(\mathbf{R}^n)$ function. w is said to satisfy a reverse Hölder inequality if for a certain $\delta > 0$ there exists a positive constant C such that for every ball $B \subset \mathbf{R}^n$

$$\left(\frac{1}{|B|}\int_B w^{1+\delta}\,dx\right)^{1/(1+\delta)} \leqslant C\frac{1}{|B|}\int_B w\,dx, \tag{1.1}$$

C and δ being independent of B. Condition (1.1) is known to imply local integrability of w^{-1}, thus providing information on the zeros of w. In fact, it turns out that if (1.1) holds, then w is an A_p-weight of Muckenhoupt for some $p \in (1, +\infty)$. By this it is meant that

$$\sup_B\left(\frac{1}{|B|}\int_B w\,dx\right)\left(\frac{1}{|B|}\int_B w^{-1/(p-1)}\,dx\right)^{p-1} \leqslant A, \tag{1.2}$$

where A is a positive constant, and the supremum is taken over the collection of all balls $B \subset \mathbf{R}^n$. From (1.2) it is clear that w cannot vanish on a set of positive measure, and that $w^{-1} \in L^{\varepsilon}_{\mathrm{loc}}$ for a certain $\varepsilon > 0$ depending on p. The standard reference for A_p-weights is the paper by Coifman and C. Fefferman **[CF]** to which we refer the reader for a proof that (1.1) implies (1.2) (the converse is also true: if a function w satisfies (1.2), then there exists a $\delta > 0$ for which (1.1) holds; see Theorem IV in **[CF]**).

Over the last few years the study of A_p-weights has received increasing attention from both real analysts and workers in partial differential equations. Several results have been obtained which show the special link existing between weights and properties of solutions to large classes of second order pde's. However, most of these results deal with positive solutions, thus it is natural to ask what happens if such a sign assumption is dropped. For instance, it is well known that if u is a positive harmonic function, then there exists a $\delta > 0$ such that u^δ is an A_2-weight, i.e., (1.2) holds for u^δ with $p = 2$. This can be seen by considering $v = \log u$ and showing

Received by the editors February 11, 1985.
1980 *Mathematics Subject Classification*. Primary 26E05, 42B25.
Key words and phrases. Real analytic function, A_p-weights of Muckenhoupt, reverse Hölder inequalities.

that $v \in$ BMO (bounded mean oscillation). Then, the lemma of John and Nirenberg [**JN**] assures that there exists a $\delta > 0$ such that $e^{\delta v} = u^{\delta} \in A_2$. However, if u is a harmonic function of variable sign, the above argument breaks down, and it is not a priori obvious that $|u|$ will satisfy an integrability condition like (1.2).

In this paper we show that given any real analytic function u in $\mathbf{R}^n$, there exists some $p > 1$ for which $|u|$ is locally an A_p-weight of Muckenhoupt. Ou proof is based on Theorem 1 which says that the square of a real analytic function satisfies locally a uniform doubling condition on balls (see (1.3)) and, moreover, that the supremum of $|u|$ on a ball is controlled by an L^2 average on a larger concentric ball (see (1.4)). In Corollary 1 we show how these estimates can be used to prove (1.1). Since for a real analytic function the devices from the theory of pde's are not readily available, we have worked mainly with spherical harmonics and used some elementary facts about holomorphic functions in $\mathbf{C}^n$ and polynomials in one variable. The main point in the proof of Theorem 1 is to get uniform bounds on the "Fourier coefficients" of u.

In what follows for $r > 0$ and $x \in \mathbf{R}^n$, we denote by $B_r(x)$ the open ball with center at x and radius r, whereas $\overline{B}_r(x)$ indicates its closure.

THEOREM 1. *For fixed δ_1, $\delta_2 > 0$ let u be a real analytic function in $B_{1+\delta_1+\delta_2}(0)$. Then there exists a positive constant K, depending on u, such that*

(i) *for $y \in \overline{B}_1(0)$, $0 < 2R \leqslant 1 + \delta_1 - |y|$,*

$$\int_{B_{2R}(y)} u^2\,dx \leqslant K \int_{B_R(y)} u^2\,dx; \tag{1.3}$$

(ii) *for $y \in \overline{B}_1(0)$, $0 < 2R \leqslant 1 + \delta_1 - |y|$,*

$$\sup_{B_R(y)} u^2 \leqslant K R^{-n} \int_{B_{2R}(y)} u^2\,dx. \tag{1.4}$$

REMARK. K in (1.3) and (1.4) cannot be taken independent of u. This can be seen considering the sequence of analytic functions in $\mathbf{R}^2$: $u_k(x, y) = \operatorname{Re} z^k$, $k \in \mathbf{N}$.

The proof of Theorem 1 will be given in §2. Here we wish to show how it can be used to prove

COROLLARY 1. *Let u be a real analytic function in the ball $B_{1+\delta_1+\delta_2}(0)$. Then there exist a $p \in (1, \infty)$ and a positive constant A, both depending on u, such that for every $y \in \overline{B}_1(0)$, and every R, with $0 < 2R < 1 + \delta_1 - |y|$, we have*

$$\left(\frac{1}{|B_R(y)|}\int_{B_R(y)} |u|\,dx\right)\left(\frac{1}{|B_R(y)|}\int_{B_R(y)} |u|^{-1/(p-1)}\,dx\right)^{p-1} \leqslant A. \tag{1.5}$$

PROOF OF COROLLARY 1. According to what has been remarked above, all we need to show is that for some $\delta > 0$ we have a reverse Hölder inequality, i.e.,

$$\left(\frac{1}{|B_R(y)|}\int_{B_R(y)} |u|^{1+\delta}\,dx\right)^{1/(1+\delta)} \leqslant C\left(\frac{1}{|B_R(y)|}\int_{B_R(y)} |u|\,dx\right) \tag{1.6}$$

with a constant $C > 0$ independent of $B_R(y)$ for y and R as in the statement of Theorem 1. Indeed, we will show that (1.6) holds with $|u|$ replaced by u^2, but this will suffice. Now from (1.4) and (1.3) in Theorem 1 we obtain

$$\sup_{B_R(y)} |u| \leqslant K_1\left(\frac{1}{|B_R(y)|}\int_{B_R(y)} u^2\,dx\right)^{1/2}, \tag{1.7}$$

where K_1 depends on u but not on $B_R(y)$. Since trivially for every $\delta > 0$

$$\left(\frac{1}{|B_R(y)|}\int_{B_R(y)} |u|^{2(1+\delta)}\,dx\right)^{1/2(1+\delta)} \leqslant \sup_{B_R(y)} |u|, \tag{1.8}$$

from (1.8) and (1.7) we obtain

$$\left(\frac{1}{|B_R(y)|}\int_{B_R(y)} |u|^{2(1+\delta)}\,dx\right)^{1/2(1+\delta)} \leqslant K_1\left(\frac{1}{|B_R(y)|}\int_{B_R(y)} |u|^2\,dx\right)^{1/2} \tag{1.9}$$

for every $\delta > 0$. This means that u^2 satisfies a reverse Hölder inequality. Therefore for some $q \in (1, \infty)$ we have (see **[CF]**)

$$\left(\frac{1}{|B_R(y)|}\int_{B_R(y)} |u|^2\,dx\right)\left(\frac{1}{|B_R(y)|}\int_{B_R(y)} |u|^{-2/(q-1)}\,dx\right)^{q-1} \leqslant A_1 \tag{1.10}$$

for every $y \in \overline{B}_1(0)$ and $B_R(y)$ with $2R < 1 + \delta_1 - |y|$. A_1, in (1.10), is a constant depending on u. Schwarz inequality and (1.10) give

$$\left(\frac{1}{|B_R(y)|}\int_{B_R(y)} |u|\,dx\right)\left(\frac{1}{|B_R(y)|}\int_{B_R(y)} |u|^{-2/(q-1)}\right)^{(q-1)/2} \leqslant (A_1)^{1/2}. \tag{1.11}$$

If we now let $p = (q+1)/2 > 1$, $(A_1)^{1/2} = A$, then we can rewrite (1.11) as

$$\left(\frac{1}{|B_R(y)|}\int_{B_R(y)} |u|\,dx\right)\left(\frac{1}{|B_R(y)|}\int_{B_R(y)} |u|^{-1/(p-1)}\,dx\right)^{p-1} \leqslant A. \tag{1.12}$$

This completes the proof of the corollary.

REMARK. We wish to emphasize that (1.3) in Theorem 1 implies that if u vanishes of infinite order at one point, then u must vanish identically.

It has been communicated to us subsequently by C. Kenig that R. Beals had sketched a proof (using different methods) of a related result in a letter to him dated April 13, 1982.

2. Proof of Theorem 1. In this section we prove some lemmas needed for the proof of Theorem 1. We assume that u is a real analytic function as in the statement of the theorem.

LEMMA 1. *There exists a $\delta < \delta_2$ and a holomorphic function $\tilde{u}$ in*

$$\Omega = \left\{ z \in \mathbf{C}^n \,\middle|\, |\operatorname{Re} z| < 1 + \delta_1 + \delta,\ |\operatorname{Im} z| < \delta \right\},$$

such that $\tilde{u}|_{\Omega \cap \mathbf{R}^n} = u$.

PROOF. For any given y such that $|y| \leqslant 1 + \delta_1$, there exists a $\rho_y > 0$, $\rho_y < \delta_2$, such that u is represented by a convergent power series on the ball $B_{\rho_y}(y)$. Since $\overline{B}_{1+\delta_1}(0)$ is compact we can find $y_1, \ldots, y_N$ in $\overline{B}_{1+\delta_1}(0)$ such that $\overline{B}_{1+\delta_1}(0) \subseteq \bigcup_{j=1}^N B_{\rho_{y_j}}(y_j)$. Let $\tilde{B}_j$, $j = 1, \ldots, N$, be the balls in $\mathbf{C}^n$ with the same radii and centers of $B_{\rho_{y_j}}(y_j)$. Define holomorphic functions $\tilde{u}_j$ on $\tilde{B}_j$ by the power series defining u. As $\tilde{u}_i = \tilde{u}_j$ on $\tilde{B}_i \cap \tilde{B}_j \cap \mathbf{R}^n$, $\tilde{u}_i = \tilde{u}_j$ on $\tilde{B}_i \cap \tilde{B}_j$. From this the lemma follows.

In the sequel we will denote by H_d the space of spherical harmonics of degree d in $\mathbf{R}^n$. a_d will indicate the dimension of H_d, and $\{\psi_{d,i}: i = 1, \ldots, a_d\}$ an orthonormal basis for H_d with respect to the natural measure $d\sigma$ on S^{n-1}.

LEMMA 2. *There exists a* $\lambda \in (0, +\infty)$, *and for every* $y \in \overline{B}_1(0)$ *there exists* $C_{d,i}^{(y)}(m)$ *such that* (*with* δ *as in Lemma* 1)

$$|C_{d,i}^{(y)}(m)| \leqslant \lambda d^n (\tfrac{3}{2}\delta)^m, \tag{2.1}$$

$$C_{d,i}^{(y)}(m) = 0 \quad \text{for } m < d, \tag{2.2}$$

and the series $\sum_{d,i,m} C_{d,i}^{(y)}(m)|x|^m \psi_{d,i}(x)$ *converges uniformly and absolutely to* $u(x+y)$ *for* $|x| \leqslant \delta/2$.

PROOF. For $z \in \mathbf{C}$, $|z| < \delta$, $|y| \leqslant 1$ define an analytic function

$$\varphi_{d,i}^{(y)}(z) = \int_{S^{n-1}} \tilde{u}(z\xi + y)\overline{\psi_{d,i}(\xi)}\, d\sigma(\xi), \tag{2.3}$$

where $\tilde{u}$ is the extension of u constructed in Lemma 1. By known properties of spherical harmonics,

$$u(x+y) = \sum_{d,i} \varphi_{d,i}^{(y)}(|x|)|x|^d \psi_{d,i}(x), \tag{2.4}$$

the convergence being uniform for x in a compact set. For $|y| \leqslant 1$, $\varphi_{d,i}^{(y)}$ has a power series expansion with coefficients $C_{d,i}^{(y)}(m)$ given by

$$C_{d,i}^{(y)}(m) = (2\pi i)^{-1} \int_{|\zeta| = 2\delta/3} \int_{S^{n-1}} \zeta^{-(m+1)} \tilde{u}(\zeta\xi + y)\overline{\psi_{d,i}(\xi)}\, d\sigma(\xi)\, d\zeta. \tag{2.5}$$

Since (see, e.g., **[SW]**)

$$\sup_{\xi \in S^{n-1}} |\psi_{d,i}(\xi)| \leqslant \sqrt{a_d} \leqslant n^n d^n \tag{2.6}$$

and u is uniformly continuous on

$$\left\{ w \in \mathbf{C}^n \,\middle|\, |\operatorname{Im} w| \leqslant \tfrac{2}{3}\delta,\ |\operatorname{Re} w| \leqslant 1 + \delta_1 + \tfrac{2}{3}\delta \right\},$$

we get (2.1) from (2.5) and (2.6). Further, the argument above shows that the series in the statement of the lemma represents $u(x+y)$ nicely. Finally, since H_d is orthogonal to every lower-degree polynomial, one sees that $C_{d,i}^{(y)}(m) = 0$ for $m < d$.

LEMMA 3. *There exist* μ, $M > 0$ *such that for all* $y \in \overline{B}_1(0)$ *we have*

$$\sup_{m \leqslant M} \sup_i |C_{d,i}^{(y)}(m)| \geqslant \mu \tag{2.7}$$

(*n.b.*, $C_{d,i}^{(y)}(m) = 0$ *for* $m < d$).

PROOF. By the expansion in Lemma 2, for each $y \in \overline{B}_1(0)$ there is some nonzero $C_{d,i}^{(y)}(m)$. By (2.5), $C_{d,i}^{(y)}(m)$ is a continuous function of y. The conclusion follows by a compactness argument.

LEMMA 4. *Fix $T > 0$. Then there exist $R_0 > 0$, $C < +\infty$, with $R_0 < \delta$, such that for $y \in \overline{B}_1(0)$ and $0 < r \leqslant R$*

$$\sum_{m > T} \sum_{d,i} |C_{d,i}^{(y)}(m)| r^m < C r^{T+3/4}. \tag{2.8}$$

PROOF. As $C_{d,i}^{(y)}(m) = 0$ for $m < d$, (2.8) follows from (2.1).

LEMMA 5. *Let $R_0 > 0$, $N \in \mathbf{N}$, and $0 < \mu \leqslant \sigma < \infty$ be two fixed numbers. Let Y be the set of polynomials in r of degree $\leqslant N$ whose coefficients C_j satisfy the constraints $\mu \leqslant |C_j| \leqslant \sigma$. Then*

$$\inf_{0 < R \leqslant R_0} \inf_{P \in Y} R^{-(2N+n)} \int_0^R |P(r)|^2 r^n \frac{dr}{r} > 0. \tag{2.9}$$

PROOF. By the equivalence of topologies on a finite-dimensional vector space, it suffices to show that

$$\sup_{0 < R \leqslant R_0} \sup_{P \in \tilde{Y}} \int_0^1 |P(r)|^2 r^n \frac{dr}{r} \Big/ \int_0^1 |P(rR)|^2 R^{-2N} r^n \frac{dr}{r} < \infty, \tag{2.10}$$

where $\tilde{Y}$ is the set of nonzero polynomials in Y. (2.10) is equivalent to

$$\sup_{0 < R \leqslant R_0} \sup_{P \in \tilde{Y}} \int_0^1 \left| P\left(\frac{r}{R}\right) R^N \right|^2 r^n \frac{dr}{r} \Big/ \int_0^1 |P(r)|^2 r^n \frac{dr}{r} < \infty. \tag{2.11}$$

Now let $P(r) = \sum_j c_j r^j$. Then

$$\sup_{0 < R \leqslant R_0} \int_0^1 \left| P\left(\frac{r}{R}\right) R^N \right|^2 r^n \frac{dr}{r} \leqslant R_0^{2N} \sum_j \frac{|C_j|^2}{(2j+n)}. \tag{2.12}$$

From (2.12), (2.11) follows, hence (2.9).

LEMMA 6. *Let Y be the set of nonzero polynomials of degree N in r. Then*

$$\sup_{0 < R} \sup_{P \in Y} \int_0^{2R} |P(r)|^2 r^n \frac{dr}{r} \Big/ \int_0^R |P(r)|^2 r^n \frac{dr}{r} < \infty. \tag{2.13}$$

PROOF. The ratio in (2.13) is

$$\int_0^2 |P(rR)|^2 r^n \frac{dr}{r} \Big/ \int_0^1 |P(rR)|^2 r^n \frac{dr}{r}.$$

Thus, the inner sup is independent of $R > 0$. Also the sup can be taken over the compact subset of those $P \in Y$ such that $\int_0^1 |P(r)|^2 r^n (dr/r) = 1$. (2.13) then follows.

LEMMA 7. *Let Y be as in Lemma 6. Then there exists $C > 0$ such that for $R > 0$, $P \in Y$,*

$$|P(0)|^2 < C R^{-n} \int_0^R |P(r)|^2 r^n \frac{dr}{r}. \tag{2.14}$$

PROOF. If $P(0) = 0$, (2.14) is trivial. Therefore we consider only the set of those $P \in Y$ such that $P(0) = 1$. By replacing $P(r)$ by $P(rR)$ we may take $R = 1$. But then $\int_0^1 |P(r)|^2 r^n (dr/r)$ is a positive definite quadratic functional, so it assumes its infimum at a unique P, with $P(0) = 1$. This yields (2.14).

We are now ready to give the

PROOF OF THEOREM 1. Let $M > 0$ be as in Lemma 3, and $R_0 < \delta$ be as in Lemma 4. For $y \in \overline{B}_1(0)$ and $x \in \overline{B}_{R_0/2}(0)$ we set

$$\begin{aligned} u_y(x) &= \sum_{m \leqslant M} \sum_{d,i} C_{d,i}^{(y)}(m) |x|^m \psi_{d,i}(x), \\ v_y(x) &= \sum_{m > M} \sum_{d,i} C_{d,i}^{(y)}(m) |x|^m \psi_{d,i}(x). \end{aligned} \tag{2.15}$$

Then for $0 < 2R \leqslant R_0$, by Lemma 2 and the Schwarz inequality, we have

$$\int_{|x| \leqslant 2R} |u(x+y)|^2 \, dx \leqslant 2\left(\int_{|x| \leqslant 2R} |u_y(x)|^2 \, dx + \int_{|x| \leqslant 2R} |v_y(x)|^2 \, dx \right). \tag{2.16}$$

We wish to show that each addend on the right-hand side of (2.16) is bounded uniformly by a constant times $\int_{|x| \leqslant R} |u_y(x)|^2 \, dx$. Such a bound for the first addend is given by Lemma 6. Lemmas 3, 4 and 5 imply that the same is true for the second addend.

Now we show that there exists a uniform $C < \infty$ such that for $0 < R \leqslant R_0/2$,

$$\int_{|x| \leqslant R} |u_y(x)|^2 \, dx \leqslant C \int_{|x| \leqslant R} |u(x+y)|^2 \, dx. \tag{2.17}$$

Indeed, this is a consequence of an argument similar to the one just given to bound the right-hand side of (2.16). Therefore, we have shown that there exists a $R_0 \in (0, \delta)$ for which a $C \in (0, +\infty)$ can be found, such that for $y \in \overline{B}_1(0)$ and $R \in (0, R_0/2]$

$$\int_{|x| < 2R} |u(x+y)|^2 \, dx \leqslant C \int_{|x| < R} |u(x+y)|^2 \, dx. \tag{2.18}$$

On the other hand, if R is such that $R_0/2 < R \leqslant 1 + \delta_1 - |y|$ we have

$$\begin{aligned} &\int_{|x| < 2R} |u(x+y)|^2 \, dx \Big/ \int_{|x| < R} |u(x+y)|^2 \, dx \\ &\quad \leqslant \int_{|x| < 1 + \delta_1} |u(x)|^2 \, dx \Big/ \inf_{y \in \overline{B}_1(0)} \int_{|x| < R_0} |u(x+y)|^2 \, dx. \end{aligned} \tag{2.19}$$

That the denominator in the right-hand side of (2.19) is nonzero follows by Lemmas 3 and 5. This completes the proof of (1.3).

For the proof of (1.4), it is enough to observe that similar arguments to those used to prove (1.3) show that for $R \in (0, R_0]$ it suffices to prove

$$|u(y)|^2 = |u_y(0)|^2 \leqslant C R^{-n} \int_{|x| < R} |u_y(x)|^2 \, dx. \tag{2.20}$$

Now the expansion in Lemma 2, and Lemma 7, immediately give (2.20). Then the argument for large R's, i.e., $R_0 \leqslant R \leqslant 1 + \delta_1 - |y|$, proceeds as in the end of the proof of (1.3).

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