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HOMOLOGY OF CLOSED GEODESICS IN CERTAIN RIEMANNIAN MANIFOLDS

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ABSTRACT. It is shown, by using the trace formula of Selberg type, that every primitive, one-dimensional homology class of a negatively curved compact locally symmetric space contains infinitely many prime closed geodesics.

0. Let M be a compact space form of a symmetric space of rank one. In this note, we prove that each homology class in $H_1(M, \mathbf{Z})$ contains infinitely many free homotopy classes of closed curves, that is, the mapping induced from the Hurewicz homomorphism $[\pi_1(M)] \to H_1(M, \mathbf{Z})$ is an ∞-to-one correspondence. One of the geometric consequences is that any *primitive* homology class contains infinitely many *prime* closed geodesics, since, as was shown by Hadamard, every nonnull homotopy class contains a closed geodesic which is automatically prime if the homology class is primitive. Here a homology class α is called primitive if α is not a nontrivial integral multiple of another homology class. If $\dim M = 2$, then one can prove the much stronger assertion that every homology class contains infinitely many prime closed geodesics (see §2).

The following theorem, which can be shown by means of a number-theoretic argument applied to the L-functions associated to length spectrum of closed geodesics (see [**1**, **4**] for proof), is somewhat related to the result.

THEOREM. *Let H be a subgroup of $H_1(M, \mathbf{Z})$ of finite index, and let α be a coset in H_1/H. If M is negatively curved, then there exist infinitely many prime closed geodesics whose homology classes are in α.*

1. The proof relies heavily on the trace formula for the heat kernel function. We shall start with a general setting. Let $\pi\colon \tilde{M} \to M$ be the universal covering of a compact Riemannian manifold M. The fundamental group $\pi_1(M)$ acts on $\tilde{M}$ in the usual way. For brevity we write Γ for $\pi_1(M)$. For an element γ in Γ, we denote by Γ_γ the centralizer of γ, and by $[\gamma]$ the conjugacy class of γ. The set of all conjugacy classes is denoted by $[\Gamma]$. Let $\rho\colon \Gamma \to U(N)$ be a unitary representation, and let E_ρ be the flat vector bundle associated to ρ. We denote by Δ_ρ the Laplacian acting on the sections of E_ρ. The fundamental solution of the heat equation on $\tilde{M}$ will be denoted by $\tilde{k}(t; \tilde{x}, \tilde{y})$. The following lemma is proved in the same way as the proof of the Selberg trace formula (see [**6**]).

LEMMA 1.

$$\operatorname{tr}(e^{-t\Delta_\rho}) = \sum_{[\gamma]\in[\Gamma]} \operatorname{tr}\rho(\gamma) \int_{\tilde{M}/\Gamma_\gamma} \tilde{k}(t; \tilde{x}, \gamma\tilde{x})\, d\tilde{x}.$$

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We set

$$f_{[\gamma]}(t) = \int_{\tilde{M}/\Gamma_\gamma} \tilde{k}(t; \tilde{x}, \gamma\tilde{x})\, d\tilde{x},$$

$$f_\alpha(t) = \sum_{[\gamma]\in\alpha} f_{[\gamma]}(t),$$

where $[\gamma] \in \alpha$ means that α is the image of $[\gamma]$ by the canonical mapping $[\Gamma] \to \Gamma/[\Gamma,\Gamma] = H_1(M, \mathbf{Z})$.

LEMMA 2. $\overline{\lim}_{t\to\infty} t^{-1} \log f_\alpha(t) = 0$.

PROOF. We denote by $\hat{H}_1$ the group of one-dimensional characters of $H_1(M, \mathbf{Z})$. Using the orthogonal relations of characters, we get

$$f_\alpha(t) = \int_{\hat{H}_1} \chi(\alpha^{-1})\, \mathrm{tr}(e^{-t\Delta_\chi})\, d\chi,$$

where $d\chi$ is the normalized Haar measure on the compact group $\hat{H}_1$. Since f_α and $\mathrm{tr}(e^{-t\Delta_\chi})$ are real valued, we find

$$f_\alpha(t) = \int_{\hat{H}_1} \mathrm{Re}\,\chi(\alpha^{-1})\, \mathrm{tr}(e^{-t\Delta_\chi})\, d\chi.$$

Note that the first eigenvalue $\lambda_1(\chi)$ of the Laplacian Δ_χ depends continuously on χ, and that $\lambda_1(\chi) = 0$ if and only if χ is the trivial character. Hence, $\lambda_1(\chi) \geq \mu$ for some positive μ on the compact set $K = \{\chi \in \hat{H}_1 : \mathrm{Re}\,\chi(\alpha^{-1}) \leq 0\}$, and for each $\chi \in K$,

$$\mathrm{tr}(e^{-t\Delta_\chi}) = e^{-\lambda_1(\chi)t} + e^{-\lambda_2(\chi)t} + \cdots = e^{-\mu t}C \quad \text{for } t \gg 0,$$

where the constant C can be chosen independently from $\chi \in K$.

We now suppose that

$$\overline{\lim_{t\to\infty}}\, t^{-1} \log f_\alpha(t) < \lambda < 0.$$

This implies that $f_\alpha(t) < e^{-t\lambda}$ for $t \gg 0$. If we set $\theta = \min(\mu, \lambda)$, then we have

$$\int_{\mathrm{Re}\,\chi(\alpha^{-1})>0} \mathrm{Re}\,\chi(\alpha^{-1})\, \mathrm{tr}(e^{-t\Delta_\chi})\, d\chi \leq Ce^{-t\theta}.$$

Take a positive ε less than θ, and let U be a neighborhood of the trivial character such that for $\chi \in U$, $\mathrm{Re}\,\chi(\alpha^{-1}) \geq c_0 > 0$, $\lambda_1(\chi) \leq \varepsilon$. We then find that

$$\int_{\mathrm{Re}\,\chi(\alpha^{-1})>0} \mathrm{Re}\,\chi(\alpha^{-1})\, \mathrm{tr}(e^{-t\Delta_\chi})\, d\chi$$
$$\geq \int_U \mathrm{Re}\,\chi(\alpha^{-1})\, \mathrm{tr}(e^{-t\Delta_\chi})\, d\chi \geq c_0 e^{-t\varepsilon} \int_U 1\, d\chi.$$

This is a contradiction.

LEMMA 3. $\overline{\lim}_{t\to\infty} t^{-1} \log f_{[e]}(t) \leq -\lambda_0(\tilde{M})$, *where* $\lambda_0(\tilde{M})$ *is the lowest bound of the spectrum of the Laplacian* $\Delta_{\tilde{M}}$ *on* $\tilde{M}$.

PROOF. This comes from the spectral representation of $\tilde{k}(t; \tilde{x}, \tilde{y})$ and the equality

$$f_{[e]}(t) = \int_M \tilde{k}(t; \tilde{x}, \tilde{x})\, d\tilde{x}.$$

In fact one easily has

$$\overline{\lim_{t\to\infty}} t^{-1} \log \tilde{k}(t; \tilde{x}, \tilde{y}) \leq -\lambda_0(\tilde{M}).$$

LEMMA 4. *If M is a locally symmetric space of negative curvature, then, for $\gamma \neq e$,*

$$\lim_{t\to\infty} t^{-1} \log f_{[\gamma]}(t) = -\lambda_0(\tilde{M}).$$

PROOF. The assertion is an immediate consequence of the trace formula established by R. Gangolli [**2**]:

$$f_{[\gamma]}(t) = (4\pi t)^{-1/2} e^{-\lambda_0(\tilde{M})t} l'_{[\gamma]} |\det(P_{[\gamma]} - I)|^{-1/2} \exp(-l^2_{[\gamma]}/4t),$$

where $l_{[\gamma]}$ is the length of a closed geodesic c with the homotopy class $[\gamma]$, $l'_{[\gamma]}$ is the length of the prime geodesic whose image coincides with c, and $P_{[\gamma]}$ is the linearized Poincaré mapping associated with c.

PROOF OF MAIN RESULT. If there exist only finitely many $[\gamma]$ with $[\gamma] \in \alpha$, then the above lemma implies

$$\overline{\lim_{t\to\infty}} t^{-1} \log f_\alpha(t) < 0,$$

which contradicts Lemma 2.

REMARK. The trace formula in Lemma 1 can be connected with a summation formula for Wiener integrals on path spaces (see [**6**]).

2. If $\dim M = 2$ and the genus of M $(= g)$ is greater than one, then $\pi_1(M)$ is isomorphic to the group generated by $2g$ elements $A_1, \ldots, A_g, B_1, \ldots, B_g$ with the single relation $\prod_{i=1}^h A_i B_i A_i^{-1} B_i^{-1} = 1$. This, in particular, implies that there exists a surjective homomorphism from $\pi_1(M)$ to the free group of rank g. We shall prove the following general theorem.

THEOREM. *Let M be a compact Riemannian manifold. If there exists a surjective homomorphism of $\pi_1(M)$ onto a nonabelian free group, then for any $\alpha \in H_1(M, \mathbf{Z})$, there exist infinitely many prime geodesics in α.*

PROOF. From the assumption, one can make a surjective homomorphism onto the free group of rank two: $\pi: \pi_1(M) \to F_2 = \langle a_1, a_2 \rangle$. Take an element γ in $\pi_1(M)$ whose image by the Hurewicz homomorphism is α, and express $\pi(\gamma)$ by a reduced word $\pi(\gamma) = x_1 \cdots x_h$, where $x_i = a_1^\varepsilon$ or a_2^ε $(\varepsilon = \pm 1)$. We set

$$\begin{aligned} a(m,n) &= x_1 \cdots x_h a_1^m a_2^n a_1^{-m} a_2^{-n}, \\ \gamma(m,n) &= \gamma\gamma_1^m \gamma_2^n \gamma_1^{-m} \gamma_2^{-n} \end{aligned} \qquad (m, n > 0)$$

where $\pi(\gamma_i) = a_i$ $(i = 1, 2)$. Note that $\gamma(m,n)$ is homologous to γ in $H_1(M, \mathbf{Z})$ for any (m,n). Without loss of generality, one may assume that the word $a(m,n)$ is reduced and cyclically reduced (if necessary, we take a_1^{-1} and a_2^{-1} instead of a_1 and a_2). See [**3**] for terminology.

We show that (i) $\gamma(m,n)$ is not a nontrivial power of another element if $n > h$, and (ii) $\gamma(m,n)$ is not conjugate to $\gamma(m',n')$ for $(m,n) \neq (m',n')$. For this, it is enough to prove the same assertion for $a(m,n)$.

Suppose that $a(m,n) = (y_1 \cdots y_k)^r$ $(r \geq 2)$, where $y_1 \cdots y_k$ is a reduced word. Since $a(m,n)$ is cyclically reduced, so is $y_1 \cdots y_k$, and

$$(y_1 \cdots y_k)^r = y_1 \cdots y_k y_1 \cdots y_k \cdots y_1 \cdots y_k$$

is reduced. This implies that the word $y_1 \cdots y_k$ has the symbol a_2^{-n} as the last n symbols, so that $(y_1 \cdots y_k)^r$ has at least rn symbols a_2^{-1}. On the other hand, $a(m,n)$ has at most $h+n$ symbols a_2^{-1}. This is a contradiction.

For (ii), we use [**3**, Theorem 3.2, p. 36], which asserts that if $a(m,n)$ is conjugate to $a(m',n')$, then $a(m,n)$ is a cyclic permutation of $a(m',n')$. Since $(m,n) \neq (m',n')$, the numbers of the symbols a_1 or a_2 in the words $a(m,n)$ and $a(m',n')$ are different. This implies (ii).

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