ABSTRACT. Assuming Rowbottom-type properties, we estimate the size of certain families of closed disjoint functions. We show that whenever \( \kappa \) is Rowbottom and \( 2^\omega < \aleph_1(\kappa) \), then \( 2^\kappa = 2^\omega \) or \( \kappa \) is the strong limit cardinal. Next we notice that every strongly inaccessible Jónsson cardinal \( \kappa \) is \( \mu \)-Rowbottom for some \( \mu < \kappa \). In turn, Shelah’s method allows us to construct a Jónsson model of cardinality \( \kappa^+ \) provided \( \kappa^{cf}(\kappa) = \kappa^+ \). We include some additional remarks.

0. Introduction. In this paper the basic set-theoretical notation is standard. We only mention that the letters \( \kappa, \lambda, \mu, \ldots \) are reserved for cardinals. All undefined notions are taken from [3].

By \( j: M \rightarrow V_\gamma \), we mean an elementary embedding of the transitive model \( M \) into the collection of all sets of rank less than the limit ordinal \( \gamma \) which moves some ordinal. We write \( (\kappa, \lambda) \rightarrow (\mu, < \nu) \) iff whenever \( f: [\kappa]^\omega \rightarrow \lambda \), there exists a set \( X \subseteq \kappa \) such that \( |X| = \mu \) and \( |f''[X]^\omega| < \nu \). A cardinal \( \kappa > \nu \) is \( \mu \)-Rowbottom iff \( (\kappa, \lambda) \rightarrow (\mu, < \nu) \) for every \( \lambda < \kappa \); \( \kappa \) is Rowbottom just in case it is \( \omega_1 \)-Rowbottom.

If \( (\kappa, \nu) \rightarrow (\kappa, < \nu) \) for some \( \nu < \kappa \), then \( \kappa \) is called Jónsson.

Background information about a relation between Jónsson cardinals and elementary embeddings can be found in [8]. Mimicking [8] we can in fact prove that the property \( (\kappa, \lambda) \rightarrow (\mu, < \nu) \) is equivalent to the existence of an elementary embedding \( j: M \rightarrow V_\gamma \) such that \( j(\delta) = \lambda \) for some \( \delta < \nu \), \( j(\tau) = \kappa \) for some \( \tau \geq \mu \) and \( \mu, \nu \in j''M \) for every \( \gamma > \kappa \) (or equivalently, for some \( \gamma > \kappa \); we may additionally demand that the model \( V_\gamma \) carries countably many relations and operations).

1. Closed disjoint functions. We say that two functions \( f \) and \( g \) on \( \lambda \) are closed disjoint iff the set \( \{ \alpha < \lambda : f(\alpha) \neq g(\alpha) \} \) contains a closed unbounded subset of \( \lambda \).

PROPOSITION 1.1. Assume \( j: M \rightarrow V_\gamma \), \( j(\delta) = \lambda \) for some \( \delta < \lambda \), \( \lambda \) is regular and \( \mu \in M \cap j''M \). Let \( h: \lambda \rightarrow \mu \). Then every family of closed disjoint functions \( f \in \prod_{\alpha < \lambda} h(\alpha) \) has less than \( j(\mu) \) elements.

PROOF. Suppose the opposite: there exist a function \( h: \lambda \rightarrow \mu \) and a family of size \( j(\mu) \) of closed disjoint functions \( f \in \prod_{\alpha < \lambda} h(\alpha) \). Let \( \delta \rightarrow \tau \) and a family \( F \subseteq M \) of size \( \mu \) of closed disjoint-in-\( M \) functions \( f \in \prod_{\alpha < \delta} h(\alpha) \). Put \( \eta = \sup j''(\delta) < \lambda \).

We shall prove that \( (j(f))(\eta) \neq (j(g))(\eta) \) for any distinct \( f, g \in F \). Let \( C \subseteq \{ \alpha < \delta : f(\alpha) \neq g(\alpha) \} \) be closed unbounded in \( \delta \) and \( C \subseteq M \). Therefore \( j''C \) is...
unbounded in $\sup j''\delta = \eta$, $j(C) \subseteq \{\alpha < \lambda: (j(f))(\alpha) \neq (j(g))(\alpha)\}$ and $j(C)$ is closed in $\lambda$. Since $j''C \in j(C)$, we have $\eta \in j(C)$.

Now we derive a contradiction, because

$$|F| = |\{(j(f))(\eta): f \in F\}| \leq (j(\bar{h}))(\eta) < \mu. \quad \square$$

In the presence of Proposition 1.1 we receive some generalizations of the known results about Chang’s Conjecture or those formulated in [7], for instance.

**Corollary 1.2.** If $\lambda$ is a regular cardinal and $(\kappa, \lambda) \rightarrow (\mu, < \lambda)$, then every family of closed disjoint functions $f: \lambda \rightarrow \eta$, where $\eta < \mu$, has less than $\kappa$ members. If in addition $\text{cf}(\mu) > \lambda$, then every family of closed disjoint functions $f: \lambda \rightarrow \mu$ has at most $\kappa$ elements.

In particular, if $\kappa$ is Rowbottom, then the above statement is true for each regular $\omega < \lambda < \kappa = \mu. \quad \square$

Let $I$ be an ideal over $\lambda$. A partial function $f$ on $\lambda$ is called an $I$-function iff $\text{dom}(f) \notin I$ (compare [3, p. 432]). Two $I$-functions $f$ and $g$ on $\lambda$ are almost disjoint iff the set $\{\alpha < \lambda: f(\alpha) = g(\alpha)\}$ has size less than $\lambda$.

The method of the proof of Proposition 1.1 yields

**Proposition 1.3.** Assume $j: M \rightarrow V_\gamma$, $j(\delta) = \lambda$ for some $\delta < \lambda$, $\lambda$ is regular, $\lambda \leq \mu \in j''M$ and $\mu^+ \in M$. Let $I$ be any ideal over $\lambda$ containing all bounded subsets of $\lambda$. Then every family of almost disjoint $I$-functions on $\lambda$ into $\mu$ has less than $j(\mu^+)$ elements.

**Proof.** Argue by contradiction. Let $j(\tau) = \mu$. By our assumption there exist an $M$-ideal $T \in M$ over $\delta$ containing all bounded subsets of $\delta$ and a family $F \in M$ of size $\mu^+$ of almost disjoint $I$-functions on $\delta$ into $\tau$. Set $\eta = \sup j''\delta < \lambda$.

For any distinct $f, g \in F$ we can choose $\beta < \delta$ such that $f(\alpha) \neq g(\alpha)$ for all $\alpha \in \text{dom}(f) \cap \text{dom}(g)$, $\alpha \geq \beta$. Thus $(j(f))(\alpha) \neq (j(g))(\alpha)$ for all $\alpha \in \text{dom}(j(f)) \cap \text{dom}(j(g))$, $\alpha \geq \eta > j(\beta)$.

Since all bounded subsets of $\lambda$ are elements of $j(T)$, there is a subfamily $G \subseteq F$ of size $\mu^+$ and an ordinal $\eta < \alpha < \lambda$ such that $\alpha \in \text{dom}(j(f))$ for all $f \in G$. However, $|G| = |\{(j(f))(\alpha): f \in G\}| \leq \mu$, which is false. $\square$

**Corollary 1.4.** If $j: M \rightarrow V_\gamma$, $j(\delta) = \lambda$ for some $\delta < \lambda$, $\lambda$ is regular and $\lambda^+ \in M$, then the ideal $T = \{x \subseteq \lambda: |x| < \lambda\}$ of bounded subsets of $\lambda$ is $j(\lambda^+)$-saturated. Hence, the ideal $T$ is $\lambda^++$-saturated, assuming $(\lambda^+, \lambda) \rightarrow (\lambda^+, < \lambda)$.

**Proof.** If $|x| = \lambda$, then the identity restricted to $x$ is an $T$-function. Now apply Proposition 1.3. $\square$

If $f$ and $g$ are ordinal-valued functions on a regular cardinal $\lambda > \omega$, then the symbol $g < f$ means that the set $\{\alpha < \lambda: g(\alpha) < f(\alpha)\}$ contains some closed unbounded subset of $\lambda$ (compare [3, p. 67]). The relation $g < f$ is well-founded and the rank $\|f\| = \sup\{\|g\| + 1: g < f\}$ of $f$ in this one is called the norm of $f$. $\|\tau\|$ denotes the norm of the constant function on $\lambda$ taking $\tau$ as the only value.

The following result can be deduced indirectly from Shelah’s work [5] using Magidor’s filters.
PROPOSITION 1.5. If \( j: M \rightarrow V, j(\delta) = \lambda \) for some \( \delta < \lambda \) and \( \lambda \) is regular, then \( \|\tau\| \leq j(\tau) \) for each \( \tau \in M \cap j''M \).

PROOF. Let \( \eta = \sup j''\delta < \lambda \). First of all we want to show that \( \|f\|^M \leq (j(f))(\eta) \) for every ordinal-valued function \( f \in M \) defined on \( \delta \). This is done by simple induction on the norm \( \|f\|^M \). If \( \|f\|^M = \beta + 1 \), then there is some function \( g \in M \) on \( \delta \) such that \( g < f \) and \( \|g\|^M = \beta \leq (j(g))(\eta) \). But the proof of Proposition 1.1 shows that \( g < f \) implies \( (j(g))(\eta) < (j(f))(\eta) \) and so \( \|f\|^M = \beta + 1 \leq (j(f))(\eta) \). The case of limit \( \|f\|^M \) is similar.

Presently, if \( \|r\| > j(\tau) \) and \( j(\alpha) = \tau \), then there exists a function \( g: \delta \rightarrow r \) such that \( \|g\| = j(\tau) \) and so in \( M \) there is a function \( f: \delta \rightarrow \alpha \) such that \( \|f\|^M = \tau \). But then \( \tau \leq (j(f))(\eta) < j(\alpha) = \tau \), a contradiction. \( \Box \)

COROLLARY 1.6. If \( \lambda \) is a regular cardinal and \( (\kappa, \lambda) \rightarrow (\mu, \lambda) \), then \( \|\mu\| \leq \kappa \). In particular, Chang’s Conjecture \((\lambda^+, \lambda) \rightarrow (\lambda, \lambda) \) implies \( \|\lambda\| = \lambda^+ \). \( \Box \)

2. Cardinal exponentiation. Using some elementary embeddings we can obtain a few inequalities in cardinal arithmetic.

LEMMA 2.1. If \( j: M \rightarrow V, j(\delta) = \lambda, \rho \in j''M, \mu = (\rho^\delta)^+ \) and \( \mu \in M \), then \( \rho^\lambda < j(\mu) \).

PROOF. Let \( j(\eta) = \rho \) and assume to the contrary that \( \rho^\lambda \geq j(\mu) \). Hence there exists some function from \( \delta^\rho \) onto \( j(\mu) \). So in \( M \) there is a function which transforms \( (\delta^\rho)^M \) onto \( \mu \). As \( \eta \leq \rho \), the contradiction \( \mu \leq (\delta^\rho)^M \leq \rho^\delta < \mu \) establishes the Lemma. \( \Box \)

COROLLARY 2.2. If \( (\kappa, \lambda) \rightarrow (\mu, \mu) \) and \( \rho^\alpha < \mu \) for all \( \alpha < \nu \), then \( \rho^\lambda < \kappa \). Therefore, if \( \kappa \) is \( \nu \)-Rowbottom and \( 2^\alpha < \kappa \) for all \( \alpha < \nu \), then \( \kappa \) is the strong limit cardinal. \( \Box \)

REMARK 2.3. If \( 2^\omega < \aleph_\omega \) and \( \aleph_\omega \) is Rowbottom, then \( \aleph_\omega \) is the strong limit cardinal. By Theorem 84 from [3] and Corollary 1.4 we can even evaluate that \( 2^{\aleph_{n+1}} = 2^{\aleph_n} \) or \( 2^{\aleph_{n+1}} < j(\aleph_{n+2}) \) for all \( n < \omega \), whenever \( j: M \rightarrow V, j(\delta) = \lambda \) and \( \alpha \in M \) and some countable ordinal is moved by \( j \). \( \Box \)

REMARK 2.4. If a cardinal \( \kappa \) is not strong limit, then the property \( (\kappa, \lambda) \rightarrow (\kappa, \lambda) \) fails for the least \( \lambda < \kappa \) such that \( 2^\lambda \geq \kappa \). \( \Box \)

The same arguments can be used for proving

LEMMA 2.5. If \( j: M \rightarrow V, j(\delta) = \lambda, \rho \in j''M, \mu = (\rho^\delta)^+ \) and \( \mu \in M \), then \( \rho^\lambda < j(\mu) \). \( \Box \)

COROLLARY 2.6. If \( (\kappa, \lambda) \rightarrow (\mu, \mu) \) and \( \alpha^\rho < \mu \) for all \( \alpha < \nu \), then \( \rho^\lambda < \kappa \). Therefore, if \( \kappa \) is \( \nu \)-Rowbottom and \( \alpha^\omega < \kappa \) for all \( \alpha < \nu \), then \( \nu^\omega < \kappa \) for all \( \lambda < \kappa \). \( \Box \)

LEMMA 2.7. If \( (\kappa, \lambda) \rightarrow (\mu, \lambda) \), \( \text{cf}(\nu) = \lambda \) and \( \alpha^\lambda < \mu \) for all \( \alpha < \nu \) and \( \mu \in M \), then \( \nu^\lambda < \kappa \). Hence \( 2^\nu < \kappa \), if \( \nu \leq \mu \) is a strong limit singular cardinal with \( \text{cf}(\nu) = \lambda \).

PROOF. Choose an elementary embedding \( j: M \rightarrow V, j(\delta) = \lambda \) for some \( \delta < \lambda \) and \( j(\tau) = \kappa \) for some \( \tau \geq \mu \) and \( \mu, \nu \in j''M \). But \( j(\nu) = \nu \) implies \( \lambda = \text{cf}(j(\nu)) = j(\text{cf}(\nu)) \geq j(\lambda) > j(\delta) = \lambda \).
Thus \( \nu \) is moved by \( j \) and so \( j \) witnesses \( (\kappa, \nu) \to (\mu, < \nu) \). Now Corollary 2.6 completes the proof. □

Galvin-Hajnal's method [2] and Corollary 1.6 allow us to formulate some bounds on \( \nu^{\text{cf}(\nu)} \) in certain cases of singular cardinals \( \nu > \mu \).

**Theorem 2.8.** Assume \( (\kappa, \lambda) \to (\mu, < \lambda) \). Let \( \lambda \leq \eta \leq \mu \) and \( \nu = \aleph_\eta \). If \( \text{cf}(\eta) = \lambda \) and \( \alpha^\lambda < \nu \) for all \( \alpha < \nu \), then \( \nu^\lambda < \aleph_\kappa \). In particular, if \( \nu \) is the strong limit singular cardinal of cofinality \( \lambda \), then \( 2^\nu < \aleph_\kappa \). □

**Corollary 2.9 (Magidor [4]).** Presuming Chang's Conjecture \( (\lambda^+, \lambda) \to (\lambda, < \lambda) \), if \( \aleph_\lambda \) is the strong limit cardinal, then \( 2^{\aleph_\lambda} < \aleph_{\lambda^+} \). □

**Corollary 2.10.** If \( \aleph_\omega \) is Rowbottom and \( \aleph_{\omega+n} \) is the strong limit cardinal for some \( n < \omega \), then \( 2^{\aleph_{\omega+n}} < \aleph_{\omega \omega} \). □

The main theorem is based on the following technical

**Claim 2.11.** Let \( X \) be the least cardinal such that \( \kappa_X > \kappa \). If \( (\kappa, \lambda) \to (\mu, < \lambda) \), \( \lambda < \mu \), \( \text{cf}(\mu) \neq \lambda \) and \( \rho^\lambda < \aleph_\lambda(\mu) \), then \( \rho^\lambda \leq \kappa \). Moreover, \( \rho^\lambda < \kappa \) unless \( \kappa \) is singular.

**Proof.** Simple arithmetic shows that \( \lambda \) is regular. Clearly, \( \nu < \lambda \leq \rho^\nu \). Assigning for each \( f \in \lambda^\rho \) the sequence \( f = \{ f \restriction \alpha: \alpha < \lambda \} \) we get the branching family \( F \) of \( \rho^\lambda \) functions from \( \lambda \) into some set of cardinality \( \rho^\nu \) (whenever \( f, g \in F \) and \( f(\beta) = g(\beta) \), then \( f(\alpha) = g(\alpha) \) for all \( \alpha < \beta \)--compare [3, p. 431]).

Let \( \rho^\nu < \sigma < \rho^\lambda \) be any regular cardinal. As \( \text{cf}(\eta) \neq \lambda \) for all cardinals \( \mu < \eta < \aleph_\lambda(\mu) \), the proof of Lemma 35.2 in [3] shows how then to construct a branching family \( G \) of \( \sigma \) functions \( f: \lambda \to \eta \) for some \( \eta < \mu \), \( \eta < \lambda \) or \( \text{cf}(\eta) = \lambda \). But Corollary 1.2 gives \( |G| < \kappa \), so we are done. □

**Lemma 2.12.** If \( \nu < \mu \), \( \text{cf}(\mu) \leq \nu \) or \( \mu \) is regular, \( (\kappa, \lambda) \to (\mu, < \lambda) \) holds for every regular \( \nu < \lambda < \mu \) and \( \kappa^- \leq \rho^\nu < \aleph_\nu(\mu) \), then \( \rho^\nu < \mu = \rho^\nu \). □

**Corollary 2.13.** Assume \( (\lambda^+, \lambda) \to (\lambda^+, < \lambda) \). Then \( 2^{\lambda^+} = \lambda \) implies \( 2^\lambda = \lambda^+ \) and \( \lambda^+ \leq 2^{< \lambda} < \aleph_\lambda \) implies \( 2^\lambda = 2^{< \lambda} \). □

**Theorem 2.14.** Let \( \nu = \sup\{\lambda < \kappa: \lambda \text{ is regular and } (\kappa, \lambda) \to (\kappa, < \lambda) \text{ fails}\} \). If \( \nu < \kappa \) and \( \kappa^- \leq 2^\nu < \aleph_\nu(\kappa) \), then \( 2^{< \kappa} = 2^\nu \). In particular, if \( \kappa \) is \( \nu^+ \)-Rowbottom and \( \kappa < 2^\nu < \aleph_\nu(\kappa) \), then \( 2^{< \kappa} = 2^\nu \).

**Proof.** If \( \nu < \kappa \) then \( (\kappa, \nu) \to (\kappa, < \nu) \) for every regular \( \nu < \lambda < \kappa \). It is easy to see that then \( \kappa \) is regular or \( \text{cf}(\kappa) \leq \nu \) (compare Lemma 2 in [8]). Applying Lemma 2.12 we finish the proof. □

**Corollary 2.15.** If \( \aleph_\omega \) is Rowbottom and \( \aleph_\omega < 2^{\aleph_n} < \aleph_{\omega+n+1} \) for some \( n < \omega \), then \( 2^{\aleph_\omega} = 2^{\aleph_n} \). □

**Remark 2.16.** If \( \kappa \) is \( \nu \)-Rowbottom and \( \kappa \leq 2^{< \nu} < \aleph_{\text{cf}(\nu)}(\kappa) \), then \( 2^{< \kappa} = 2^{< \nu} \). □

**Question 2.17.** Under the notation of Theorem 2.14, is \( \nu < \kappa \) whenever \( \kappa \) is Jónsson?
3. The first critical point. The least ordinal moved by an elementary embedding \( j: M \to V_\gamma \) is regular in \( M \). Thus the correspondence between Rowbottom-type properties and elementary embeddings shows that the first cardinal \( \nu \leq \mu \) such that \( (\kappa, \nu) \to (\mu, < \nu) \) is regular. We shall prove that such \( \nu \) cannot be strongly inaccessible, whenever \( \mu = \kappa \).

**Lemma 3.1.** Assume that \( (\kappa, \nu) \to (\kappa, < \nu) \) and \( \nu < \kappa \) is a limit cardinal. Let \( \lambda \leq \kappa \) be the least cardinal such that \( \lambda > \nu \) and the property \( (\kappa, \lambda) \to (\kappa, < \lambda) \) fails. If \( \nu^{\text{cf}}(\nu) < \lambda \), then there exists \( \rho < \nu \) such that \( (\kappa, \nu) \to (\kappa, < \rho) \).

**Proof.** Set \( A = \{ \sigma < \nu : (\kappa, \sigma) \to (\kappa, < \sigma) \} \) and \( \eta = \sup A \). If \( \eta < \nu \), then our statement is true for \( \rho = \eta^+ \). If this were false, there would be some strictly increasing sequence \( (\sigma_\alpha : \alpha < \text{cf}(\nu)) \) of elements of \( A \), cofinal in \( \nu \). For each \( \alpha < \text{cf}(\nu) \) we can find a partition \( f_\alpha : \kappa^{<\omega} \to \sigma_\alpha \) such that \( |f_\alpha''[X]^{<\omega}| = \sigma_\alpha \) for every \( X \subseteq \kappa \) of size \( \kappa \) (a counterexample for \( (\kappa, \sigma_\alpha) \to (\kappa, < \sigma_\alpha) \)).

Put \( B = \prod_{\alpha < \text{cf}(\nu)} \sigma_\alpha \) and define \( g : [\kappa]^{<\omega} \to B \) setting \( g(s) = (f_\alpha(s) : \alpha < \text{cf}(\nu)) \) for each finite subset \( s \subseteq \kappa \). As \( |B| = \nu^{\text{cf}}(\nu) < \lambda \), the definition of \( \lambda \) supplies a set \( X \subseteq \kappa \) such that \( |X| = \kappa \) and \( |g''[X]^{<\omega}| < \nu \). On the other hand, \( |g''[X]^{<\omega}| \geq \sup_{\alpha < \text{cf}(\nu)} |f''_\alpha[X]^{<\omega}| \geq \nu \) by definition of \( g \). This contradiction establishes the Lemma. \( \square \)

**Corollary 3.2.** If \( \nu < \kappa \) is a strong limit cardinal and \( (\kappa, \nu) \to (\kappa, < \nu) \), then \( (\kappa, \nu) \to (\kappa, < \rho) \) for some \( \rho < \nu \). Thus the least cardinal \( \lambda < \kappa \) such that \( (\kappa, \lambda) \to (\kappa, < \lambda) \) cannot be strongly inaccessible.

**Proof.** Let \( \lambda \leq \kappa \) be the least cardinal such that \( \lambda > \nu \) and \( (\kappa, \lambda) \to (\kappa, < \lambda) \) fails. Observe that \( \lambda \) is \( \nu \)-Rowbottom. Since \( 2^{<\nu} = \nu \), the cardinal \( \lambda \) is strong limit by Corollary 2.2. Now \( \nu^{\text{cf}}(\nu) = 2^\nu < \lambda \) and our claim follows from Lemma 3.1. \( \square \)

**Question 3.3.** Is the least \( \lambda < \kappa \) such that \( (\kappa, \lambda) \to (\kappa, < \lambda) \) always a successor cardinal?

**Theorem 3.4.** Every strongly inaccessible Jónsson cardinal \( \kappa \) is \( \rho \)-Rowbottom for some \( \rho < \kappa \).

**Proof.** Let \( \lambda < \kappa \) be any regular cardinal such that \( (\kappa, \lambda) \to (\kappa, < \lambda) \). The set \( S = \{ \nu < \kappa : \text{cf}(\nu) = \lambda \text{ and } \nu \text{ is strong limit} \} \) is stationary in \( \kappa \). Lemma 2 from [8] shows that \( (\kappa, \nu) \to (\kappa, < \nu) \) for every \( \nu \in S \). By Corollary 3.2, for each \( \nu \in S \) we can choose some \( \rho_\nu < \nu \) so that \( (\kappa, \nu) \to (\kappa, < \rho_\nu) \). By Fodor’s Theorem there exist some fixed \( \rho < \kappa \) and a stationary subset \( T \subseteq S \) such that \( (\kappa, \nu) \to (\kappa, < \rho) \) for each \( \nu \in T \). This means that \( \kappa \) is \( \rho \)-Rowbottom, since \( T \) is unbounded in \( \kappa \). \( \square \)

**Question 3.5.** May we erase the word “strongly” from the above Theorem?

4. Jónsson models and successor cardinals. We showed in [8] that a successor cardinal \( \kappa^+ \) is not Jónsson, whenever \( \kappa \) is regular. Alternatively, \( \kappa^+ \) is not Jónsson under \( 2^\kappa = \kappa^+ \) [1]. Shelah’s method from [6] enables us to weaken this presumption.

We say that a regular cardinal \( \mu \) is a possible scale for the sequence \( \langle \kappa_i : i < \lambda \rangle \) of cardinals iff there exists a sequence \( \langle f_\alpha : \alpha < \mu \rangle \) of functions on \( \lambda \) such that

(i) \( f_\alpha \in \prod_{i < \lambda} \kappa_i \) for all \( \alpha < \mu \),
(ii) \( \{|i < \lambda : f_\alpha(i) \geq f_\beta(i)\} \subseteq \lambda \) for all \( \alpha < \beta < \mu \). (We shall write \( f_\alpha < f_\beta \).)
(iii) For every \( f \in \prod_{i < \lambda} \kappa_i \) there exists \( \alpha < \mu \) such that \( \{|i < \lambda : f(i) \leq f_\alpha(i)\} = \lambda \) (compare [6]).
LEMMA 4.1 (SHELAH [6]). Let \( j: M \rightarrow V_\gamma \) and \( N = j''M \). If \( \mu \) is a possible scale for the sequence \( \langle \kappa_i: i < \lambda \rangle \in N \) of regular cardinals, \( \lambda + 1 \subseteq N \) and \( j(\mu) = \mu \), then \( \langle i < \lambda: j(\kappa_i) = \kappa_i \rangle = \lambda \).

PROOF. Set \( A = \{i < \lambda: |N \cap \kappa_i| < \kappa_i\} \) and assume to the contrary that \( |\lambda \setminus A| < \lambda \). As \( N \) is the elementary substructure of \( V_\gamma \), some sequence \( \langle f_\alpha: \alpha < \mu \rangle \in N \) exemplifies that \( \mu \) is a possible scale for \( \langle \kappa_i: i < \lambda \rangle \). For each \( i \in A \) the subset \( B_i = \{f_\alpha(i): \alpha \in N \cap \mu\} \) of \( N \cap \kappa_i \) has cardinality less than \( \kappa_i \), so we may choose \( \sup B_i < f(i) < \kappa_i \) by regularity of \( \kappa_i \). Accepting \( f(i) = 0 \) for \( i \in \lambda \setminus A \), we have \( f_\alpha < f \) for every \( \alpha \in N \cap \mu \). As the relation \( < \) is transitive and the set \( N \cap \mu \) is cofinal in \( \mu \), \( f_\alpha < f \) for every \( \alpha < \mu \), which is impossible. \( \square \)

THEOREM 4.2. If \( \kappa^{\text{cf}(\kappa)} = \kappa^+ \), then \( \kappa^+ \) is not Jónsson.

PROOF. By our first remark in this item we may assume that \( \lambda = \text{cf}(\kappa) < \kappa \).

Suppose that \( \kappa^+ \) is Jónsson and pick an elementary embedding \( j: M \rightarrow V_{\kappa^++\omega} \) such that \( j(\alpha) = \alpha \) for all \( \alpha \leq \lambda \), \( j(\nu) > \nu \) for some \( \nu < \kappa \) and \( j(\kappa^+) = \kappa^+ \) (see the proof of Theorem 1 in [8]). Then there exists some strictly increasing sequence \( \langle \kappa_i: i < \lambda \rangle \in j''M \) of cardinals, cofinal in \( \kappa^+ \), with \( \kappa_0 \geq \nu \).

Cantor's diagonalization method shows that every family of \( \kappa \) functions \( f \in \prod_{i < \lambda} \kappa_i^{++} \) has an upper bound in the relation \( < \). Since \( |\prod_{i < \lambda} \kappa_i^{++}| = \kappa^\lambda = \kappa^+ \), \( \kappa^+ \) is the only possible scale for \( \langle \kappa_i^{++}: i < \lambda \rangle \). Now, by Lemma 4.1, the set \( A = \{i < \lambda: j(\kappa_i^{++}) = \kappa_i^{++}\} \) is unbounded in \( \lambda \). But each \( \kappa_i^{++} \), where \( i \in A \), is Jónsson, contradicting [8]. \( \square \)

COROLLARY 4.3 (SHELAH [6]). If \( 2^{\aleph_\alpha} = \aleph_{\alpha+\gamma+1} \) and \( \text{cf}(\gamma) < \aleph_{\alpha+1} \), then \( 2^{\aleph_\alpha} \) cannot be Jónsson. \( \square \)

With a slight modification, a similar argument can be used for

LEMMA 4.4. If \( \kappa^+ \) is Jónsson, \( \lambda = \text{cf}(\kappa) > \omega \) and \( \rho^\lambda < \kappa \) for \( \rho < \kappa \), then the set \( \{\rho < \kappa: \rho^+ \) is Jónsson\} contains some closed unbounded subset of \( \kappa \).

PROOF. Let \( j: M \rightarrow V_{\kappa^++\omega} \), \( j(\alpha) = \alpha \) for all \( \alpha \leq \lambda \), \( j(\nu) > \nu \) for some \( \nu < \kappa \) and \( j(\kappa^+) = \kappa^+ \). Choose a strictly increasing continuous sequence \( \langle \rho_i: i < \lambda \rangle \in j''M \) of cardinals, cofinal in \( \kappa^+ \), with \( \rho_0 \geq \nu \). Suppose by way of contradiction that the set \( S = \{i < \lambda: \rho_i^+ \) is not Jónsson\} is stationary in \( \lambda \).

Set \( \kappa_i = \rho_i^+ \) for \( i \in S \) and \( \kappa_i = \rho_i^{++} \) for \( i \in \lambda \setminus S \). Thus no \( \kappa_i \) is Jónsson. Since \( S \) is stationary, by an analogue of Lemma 8.5 stated in [3], every family of almost disjoint functions \( f \in \prod_{i < \lambda} \kappa_i \) has at most \( \kappa^+ \) elements. But every family of \( \kappa \) functions \( f \in \prod_{i < \lambda} \kappa_i \) has an upper bound in the relation \( < \). Thus \( \kappa^+ \) is the only possible scale for \( \langle \kappa_i: i < \lambda \rangle \). Now each element of the set \( \{\kappa_i: i < \lambda \) and \( \kappa_i \neq \kappa_i \} \) is Jónsson, contrary to our choice. \( \square \)

REMARK 4.5. We recall another result of Shelah from [6] which can be formulated as follows: If \( \rho^{\text{cf}(\kappa)} < \kappa \) for all \( \rho < \kappa \) and \( \kappa^+ \) is Jónsson, then the set \( \{\lambda < \kappa: \lambda \) is a regular Jónsson cardinal\} is unbounded in \( \kappa \). Hence, if \( \lambda \) is arbitrary and \( \lambda^\omega = \aleph_\alpha \), then \( \aleph_{\alpha+\omega+1} \) cannot be Jónsson. \( \square \)

We can also leave out one assumption in Shelah's result [6].

LEMMA 4.6. If \( (\lambda^+)^\omega = \lambda^+ \) for all singular cardinals \( \lambda \), then no successor cardinal is Jónsson.
PROOF. Suppose to the contrary that $\kappa^+$ is the least successor cardinal which is Jónsson. There are now two cases:

Case I: $\text{cf}(\kappa) = \omega$. Then $\kappa^{\text{cf}(\kappa)} \leq (\kappa^+)^\omega = \kappa^+$ and a contradiction follows from Theorem 4.2.

Case II: $\kappa > \text{cf}(\kappa) > \omega$. First, collapse $\text{cf}(\kappa)$ onto $\omega_1$ using the notion of forcing $P = \{ p : p \text{ is a function with } \text{dom}(p) \in \omega_1 \text{ and } \text{ran}(p) \subseteq \text{cf}(\kappa) \}$ ordered by inclusion.

Let $G$ be any generic filter on $P$. Since $|P| = \text{cf}(\kappa)^\omega < \kappa$, it follows from [8] that $\kappa^+$ remains Jónsson in the forcing extension $V[G]$ of a ground model $V$. Clearly, since $P$ is $\omega_1$-closed, $\omega_1$ is preserved and $\text{cf}(\kappa) = \omega_1$ in $V[G]$. Moreover, $|P|$ is collapsed onto $\omega_1$ and every cardinal $\lambda > |P|$ in $V$ remains a cardinal in $V[G]$. The equality $(\lambda^+)^\omega = \lambda^+$ is also true in $V[G]$ for every singular cardinal $\lambda$.

From now on work in $V[G]$. Let $j : M \rightarrow V_\gamma$ witness that $\kappa^+$ is Jónsson. Pick some strictly increasing continuous sequence $(\kappa_i : i < \omega_1) \in j''M$ of cardinals with cofinality $\omega$, cofinal in $\kappa$. Since $|\prod_{i < 1} \kappa_i^+| \leq (\kappa^+_1)^\omega = \kappa_1^+$ for all $1 < \omega_1$, by Claim 13 from [6] the successor $\kappa^+$ is a possible scale for $(\kappa_i^+: i < \omega_1)$. It follows from Lemma 4.1 that the set $A = \{ i : \omega_1 : j(\kappa_i^+) = \kappa_i^+ \}$ is unbounded in $\omega_1$. But if $i \in A$ and $\kappa_i^+$ is greater than the first ordinal moved by $j$, then $\kappa_i^+$ is Jónsson, which is a contradiction because $\text{cf}(\kappa_i) = \omega$ and $(\kappa_i)^\omega = \kappa_i^+$. \qed

The same proof shows

LEMMA 4.7. Assume the Singular Cardinals Hypothesis. Then $2^\omega < \kappa$ implies that $\kappa^+$ is not Jónsson. \qed

LEMMA 4.8. No successor cardinal above a compact cardinal is Jónsson.

PROOF. Solovay showed that the singular cardinals hypothesis holds above the least compact cardinal (see [3, p. 405]). Now proceed as in the proof of Lemma 4.6. \qed

REFERENCES


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