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ROWBOTTOM-TYPE PROPERTIES AND A CARDINAL ARITHMETIC

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ABSTRACT. Assuming Rowbottom-type properties, we estimate the size of certain families of closed disjoint functions. We show that whenever κ is Rowbottom and $2^\omega < \aleph_{\omega_1}(\kappa)$, then $2^{<\kappa} = 2^\omega$ or κ is the strong limit cardinal. Next we notice that every strongly inaccessible Jónsson cardinal κ is ν-Rowbottom for some $\nu < \kappa$. In turn, Shelah's method allows us to construct a Jónsson model of cardinality κ^+ provided $\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$. We include some additional remarks.

0. Introduction. In this paper the basic set-theoretical notation is standard. We only mention that the letters $\kappa, \lambda, \mu, \ldots$ are reserved for cardinals. All undefined notions are taken from [**3**].

By $j: M \to V_\gamma$ we mean an elementary embedding of the transitive model M into the collection of all sets of rank less than the limit ordinal γ which moves some ordinal. We write $(\kappa, \lambda) \to (\mu, < \nu)$ iff whenever $f: [\kappa]^{<\omega} \to \lambda$, there exists a set $X \subseteq \kappa$ such that $|X| = \mu$ and $|f''[X]^{<\omega}| < \nu$. A cardinal $\kappa > \nu$ is ν-Rowbottom iff $(\kappa, \lambda) \to (\kappa, < \nu)$ for every $\lambda < \kappa$; κ is Rowbottom just in case it is ω_1-Rowbottom. If $(\kappa, \nu) \to (\kappa, < \nu)$ for some $\nu < \kappa$, then κ is called Jónsson.

Background information about a relation between Jónsson cardinals and elementary embeddings can be found in [**8**]. Mimicking [**8**] we can in fact prove that the property $(\kappa, \lambda) \to (\mu, < \nu)$ is equivalent to the existence of an elementary embedding $j: M \to V_\gamma$ such that $j(\delta) = \lambda$ for some $\delta < \nu$, $j(\tau) = \kappa$ for some $\tau \geq \mu$ and $\mu, \nu \in j''M$ for every $\gamma > \kappa$ (or equivalently, for some $\gamma > \kappa$; we may additionally demand that the model V_γ carries countably many relations and operations).

1. Closed disjoint functions. We say that two functions f and g on λ are closed disjoint iff the set $\{\alpha < \lambda: f(\alpha) \neq g(\alpha)\}$ contains a closed unbounded subset of λ.

PROPOSITION 1.1. *Assume $j: M \to V_\gamma$, $j(\delta) = \lambda$ for some $\delta < \lambda$, λ is regular and $\mu \in M \cap j''M$. Let $h: \lambda \to \mu$. Then every family of closed disjoint functions $f \in \prod_{\alpha<\lambda} h(\alpha)$ has less than $j(\mu)$ elements.*

PROOF. Suppose the opposite: there exist a function $h: \lambda \to \mu$ and a family of size $j(\mu)$ of closed disjoint functions $f \in \prod_{\alpha<\lambda} h(\alpha)$. Let $j(\tau) = \mu$. By elementarity and absoluteness we find in M a function $\overline{h}: \delta \to \tau$ and a family $F \in M$ of size μ of closed disjoint-in-M functions $f \in \prod_{\alpha<\delta} \overline{h}(\alpha)$. Put $\eta = \sup j''\delta < \lambda$.

We shall prove that $(j(f))(\eta) \neq (j(g))(\eta)$ for any distinct $f, g \in F$. Let $C \subseteq \{\alpha < \delta: f(\alpha) \neq g(\alpha)\}$ be closed unbounded in δ and $C \in M$. Therefore $j''C$ is

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unbounded in $\sup j''\delta = \eta$, $j(C) \subseteq \{\alpha < \lambda : (j(f))(\alpha) \neq (j(g))(\alpha)\}$ and $j(C)$ is closed in λ. Since $j''C \in j(C)$, we have $\eta \in j(C)$.

Now we derive a contradiction, because

$$|F| = |\{(j(f))(\eta) : f \in F\}| \leq (j(\bar{h}))(\eta) < \mu. \quad \square$$

In the presence of Proposition 1.1 we receive some generalizations of the known results about Chang's Conjecture or those formulated in [**7**], for instance.

COROLLARY 1.2. *If λ is a regular cardinal and $(\kappa, \lambda) \to (\mu, < \lambda)$, then every family of closed disjoint functions $f : \lambda \to \eta$, where $\eta < \mu$, has less than κ members. If in addition* $\mathrm{cf}(\mu) > \lambda$, *then every family of closed disjoint functions $f : \lambda \to \mu$ has at most κ elements.*

In particular, if κ is Rowbottom, then the above statement is true for each regular $\omega < \lambda < \kappa = \mu$. $\square$

Let I be an ideal over λ. A partial function f on λ is called an I-function iff $\mathrm{dom}(f) \notin I$ (compare [**3**, p. 432]). Two I-functions f and g on λ are almost disjoint iff the set $\{\alpha < \lambda : f(\alpha) = g(\alpha)\}$ has size less than λ.

The method of the proof of Proposition 1.1 yields

PROPOSITION 1.3. *Assume $j : M \to V_\gamma$, $j(\delta) = \lambda$ for some $\delta < \lambda$, λ is regular, $\lambda \leq \mu \in j''M$ and $\mu^+ \in M$. Let I be any ideal over λ containing all bounded subsets of λ. Then every family of almost disjoint I-functions on λ into μ has less than $j(\mu^+)$ elements.*

PROOF. Argue by contradiction. Let $j(\tau) = \mu$. By our assumption there exist an M-ideal $\bar{I} \in M$ over δ containing all bounded subsets of δ and a family $F \in M$ of size μ^+ of almost disjoint $\bar{I}$-functions on δ into τ. Set $\eta = \sup j''\delta < \lambda$.

For any distinct $f, g \in F$ we can choose $\beta < \delta$ such that $f(\alpha) \neq g(\alpha)$ for all $\alpha \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$, $\alpha \geq \beta$. Thus $(j(f))(\alpha) \neq (j(g))(\alpha)$ for all $\alpha \in \mathrm{dom}(j(f)) \cap \mathrm{dom}(j(g))$, $\alpha \geq \eta > j(\beta)$.

Since all bounded subsets of λ are elements of $j(\bar{I})$, there is a subfamily $G \subseteq F$ of size μ^+ and an ordinal $\eta \leq \alpha < \lambda$ such that $\alpha \in \mathrm{dom}(j(f))$ for all $f \in G$. However, $|G| = |\{(j(f))(\alpha) : f \in G\}| \leq \mu$, which is false. $\square$

COROLLARY 1.4. *If $j : M \to V_\gamma$, $j(\delta) = \lambda$ for some $\delta < \lambda$, λ is regular and $\lambda^+ \in M$, then the ideal $\bar{I} = \{x \subseteq \lambda : |x| < \lambda\}$ of bounded subsets of λ is $j(\lambda^+)$-saturated. Hence, the ideal $\bar{I}$ is λ^{++}-saturated, assuming $(\lambda^{++}, \lambda) \to (\lambda^+, < \lambda)$.*

PROOF. If $|x| = \lambda$, then the identity restricted to x is an $\bar{I}$-function. Now apply Proposition 1.3. $\square$

If f and g are ordinal-valued functions on a regular cardinal $\lambda > \omega$, then the symbol $g < f$ means that the set $\{\alpha < \lambda : g(\alpha) < f(\alpha)\}$ contains some closed unbounded subset of λ (compare [**3**, p. 67]). The relation $g < f$ is well-founded and the rank $\|f\| = \sup\{\|g\| + 1 : g < f\}$ of f in this one is called the norm of f. $\|\tau\|$ denotes the norm of the constant function on λ taking τ as the only value.

The following result can be deduced indirectly from Shelah's work [**5**] using Magidor's filters.

PROPOSITION 1.5. *If $j: M \to V_\gamma$, $j(\delta) = \lambda$ for some $\delta < \lambda$ and λ is regular, then $\|\tau\| \leq j(\tau)$ for each $\tau \in M \cap j''M$.*

PROOF. Let $\eta = \sup j''\delta < \lambda$. First of all we want to show that $\|f\|^M \leq (j(f))(\eta)$ for every ordinal-valued function $f \in M$ defined on δ. This is done by simple induction on the norm $\|f\|^M$. If $\|f\|^M = \beta + 1$, then there is some function $g \in M$ on δ such that $g < f$ and $\|g\|^M = \beta \leq (j(g))(\eta)$. But the proof of Proposition 1.1 shows that $g < f$ implies $(j(g))(\eta) < (j(f))(\eta)$ and so $\|f\|^M = \beta + 1 \leq (j(f))(\eta)$. The case of limit $\|f\|^M$ is similar.

Presently, if $\|\tau\| > j(\tau)$ and $j(\alpha) = \tau$, then there exists a function $g: \lambda \to \tau$ such that $\|g\| = j(\tau)$ and so in M there is a function $f: \delta \to \alpha$ such that $\|f\|^M = \tau$. But then $\tau \leq (j(f))(\eta) < j(\alpha) = \tau$, a contradiction. □

COROLLARY 1.6. *If λ is a regular cardinal and $(\kappa, \lambda) \to (\mu, < \lambda)$, then $\|\mu\| \leq \kappa$. In particular, Chang's Conjecture $(\lambda^+, \lambda) \to (\lambda, , < \lambda)$ implies $\|\lambda\| = \lambda^+$.* □

2. Cardinal exponentiation. Using some elementary embeddings we can obtain a few inequalities in cardinal arithmetic.

LEMMA 2.1. *If $j: M \to V_\gamma$, $j(\delta) = \lambda$, $\rho \in j''M$, $\mu = (\rho^\delta)^+$ and $\mu \in M$, then $\rho^\lambda < j(\mu)$.*

PROOF. Let $j(\eta) = \rho$ and assume to the contrary that $\rho^\lambda \geq j(\mu)$. Hence there exists some function from ${}^\lambda\rho$ onto $j(\mu)$. So in M there is a function which transforms $({}^\delta\eta)^M$ onto μ. As $\eta \leq \rho$, the contradiction $\mu \leq |({}^\delta\eta)^M| \leq \rho^\delta < \mu$ establishes the Lemma. □

COROLLARY 2.2. *If $(\kappa, \lambda) \to (\mu, < \nu)$ and $\rho^\alpha < \mu$ for all $\alpha < \nu$, then $\rho^\lambda < \kappa$. Therefore, if κ is ν-Rowbottom and $2^\alpha < \kappa$ for all $\alpha < \nu$, then κ is the strong limit cardinal.* □

REMARK 2.3. If $2^\omega < \aleph_\omega$ and $\aleph_\omega$ is Rowbottom, then $\aleph_\omega$ is the strong limit cardinal. By Theorem 84 from [**3**] and Corollary 1.4 we can even evaluate that $2^{\aleph_{n+1}} = 2^{\aleph_n}$ or $2^{\aleph_{n+1}} < j(\aleph_{n+2})$ for all $n < \omega$, whenever $j: M \to V_\gamma$ is an arbitrary elementary embedding such that $j(\aleph_\omega) = \aleph_\omega$ and some countable ordinal is moved by j. □

REMARK 2.4. If a cardinal κ is not strong limit, then the property $(\kappa, \lambda) \to (\kappa, < \lambda)$ fails for the least $\lambda < \kappa$ such that $2^\lambda \geq \kappa$. □

The same arguments can be used for proving

LEMMA 2.5. *If $j: M \to V_\gamma$, $j(\delta) = \lambda$, $\rho \in j''M$, $\mu = (\delta^\rho)^+$ and $\mu \in M$, then $\lambda^\rho < j(\mu)$.* □

COROLLARY 2.6. *If $(\kappa, \lambda) \to (\mu, < \nu)$ and $\alpha^\rho < \mu$ for all $\alpha < \nu$, then $\lambda^\rho < \kappa$. Therefore, if κ is ν-Rowbottom and $\alpha^\omega < \kappa$ for all $\alpha < \nu$, then $\lambda^\omega < \kappa$ for all $\lambda < \kappa$.* □

LEMMA 2.7. *If $(\kappa, \lambda) \to (\mu, < \lambda)$, $\mathrm{cf}(\nu) = \lambda$ and $\alpha^\lambda < \mu$ for all $\alpha < \nu \leq \mu$, then $\nu^\lambda < \kappa$. Hence $2^\nu < \kappa$, if $\nu \leq \mu$ is a strong limit singular cardinal with $\mathrm{cf}(\nu) = \lambda$.*

PROOF. Choose an elementary embedding $j: M \to V_{\kappa+\omega}$ such that $j(\delta) = \lambda$ for some $\delta < \lambda$, $j(\tau) = \kappa$ for some $\tau \geq \mu$ and $\mu, \nu \in j''M$. But $j(\nu) = \nu$ implies

$$\lambda = \mathrm{cf}(j(\nu)) = j(\mathrm{cf}^M(\nu)) \geq j(\lambda) > j(\delta) = \lambda.$$

Thus ν is moved by j and so j witnesses $(\kappa, \nu) \to (\mu, < \nu)$. Now Corollary 2.6 completes the proof. $\square$

Galvin-Hajnal's method [**2**] and Corollary 1.6 allow us to formulate some bounds on $\nu^{\mathrm{cf}(\nu)}$ in certain cases of singular cardinals $\nu > \mu$.

THEOREM 2.8. *Assume $(\kappa, \lambda) \to (\mu, < \lambda)$. Let $\lambda \leq \eta \leq \mu$ and $\nu = \aleph_\eta$. If $\mathrm{cf}(\eta) = \lambda$ and $\alpha^\lambda < \nu$ for all $\alpha < \nu$, then $\nu^\lambda < \aleph_\kappa$. In particular, if ν is the strong limit singular cardinal of cofinality λ, then $2^\nu < \aleph_\kappa$.* $\square$

COROLLARY 2.9 (MAGIDOR [**4**]). *Presuming Chang's Conjecture $(\lambda^+, \lambda) \to (\lambda, < \lambda)$, if $\aleph_\lambda$ is the strong limit cardinal, then $2^{\aleph_\lambda} < \aleph_{\lambda^+}$.* $\square$

COROLLARY 2.10. *If $\aleph_\omega$ is Rowbottom and $\aleph_{\omega_n}$ is the strong limit cardinal for some $n < \omega$, then $2^{\aleph_{\omega_n}} < \aleph_{\omega_\omega}$.* $\square$

The main theorem is based on the following technical

CLAIM 2.11. *Let λ be the least cardinal such that $\rho^\lambda > \rho^\nu$. If $(\kappa, \lambda) \to (\mu, < \lambda)$, $\lambda < \mu$, $\mathrm{cf}(\mu) \neq \lambda$ and $\rho^\nu < \aleph_\lambda(\mu)$, then $\rho^\lambda \leq \kappa$. Moreover, $\rho^\lambda < \kappa$ unless κ is singular.*

PROOF. Simple arithmetic shows that λ is regular. Clearly, $\nu < \lambda \leq \rho^\nu$. Assigning for each $f \in {}^\lambda\rho$ the sequence $\overline{f} = \langle f \restriction \alpha : \alpha < \lambda \rangle$ we get the branching family F of ρ^λ functions from λ into some set of cardinality ρ^ν (whenever $\overline{f}, \overline{g} \in F$ and $\overline{f}(\beta) = \overline{g}(\beta)$, then $\overline{f}(\alpha) = \overline{g}(\alpha)$ for all $\alpha \leq \beta$—compare [**3**, p. 431]).

Let $\rho^\nu < \sigma \leq \rho^\lambda$ be any regular cardinal. As $\mathrm{cf}(\eta) \neq \lambda$ for all cardinals $\mu \leq \eta < \aleph_\lambda(\mu)$, the proof of Lemma 35.2 in [**3**] shows how then to construct a branching family G of σ functions $f: \lambda \to \eta$ for some $\eta < \mu$, $\eta \leq \lambda$ or $\mathrm{cf}(\eta) = \lambda$. But Corollary 1.2 gives $|G| < \kappa$, so we are done. $\square$

LEMMA 2.12. *If $\nu < \mu$, $\mathrm{cf}(\mu) \leq \nu$ or μ is regular, $(\kappa, \lambda) \to (\mu, < \lambda)$ holds for every regular $\nu < \lambda < \mu$ and $\kappa^- \leq \rho^\nu < \aleph_{\nu^+}(\mu)$, then $\rho^{<\mu} = \rho^\nu$.* $\square$

COROLLARY 2.13. *Assume $(\lambda^{++}, \lambda) \to (\lambda^+, < \lambda)$. Then $2^{<\lambda} = \lambda$ implies $2^\lambda = \lambda^+$ and $\lambda^+ \leq 2^{<\lambda} < \aleph_\lambda$ implies $2^\lambda = 2^{<\lambda}$.* $\square$

THEOREM 2.14. *Let $\nu = \sup\{\lambda < \kappa : \lambda$ is regular and $(\kappa, \lambda) \to (\kappa, < \lambda)$ fails$\}$. If $\nu < \kappa$ and $\kappa^- \leq 2^\nu < \aleph_{\nu^+}(\kappa)$, then $2^{<\kappa} = 2^\nu$. In particular, if κ is ν^+-Rowbottom and $\kappa \leq 2^\nu < \aleph_{\nu^+}(\kappa)$, then $2^{<\kappa} = 2^\nu$.*

PROOF. If $\nu < \kappa$ then $(\kappa, \lambda) \to (\kappa, < \lambda)$ for every regular $\nu < \lambda < \kappa$. It is easy to see that then κ is regular or $\mathrm{cf}(\kappa) \leq \nu$ (compare Lemma 2 in [**8**]). Applying Lemma 2.12 we finish the proof. $\square$

COROLLARY 2.15. *If $\aleph_\omega$ is Rowbottom and $\aleph_\omega < 2^{\aleph_n} < \aleph_{\omega_{n+1}}$ for some $n < \omega$, then $2^{\aleph_\omega} = 2^{\aleph_n}$.* $\square$

REMARK 2.16. If κ is ν-Rowbottom and $\kappa \leq 2^{<\nu} < \aleph_{\mathrm{cf}(\nu)}(\kappa)$, then $2^{<\kappa} = 2^{<\nu}$. $\square$

QUESTION 2.17. Under the notation of Theorem 2.14, is $\nu < \kappa$ whenever κ is Jónsson?

3. The first critical point. The least ordinal moved by an elementary embedding $j: M \to V_\gamma$ is regular in M. Thus the correspondence between Rowbottom-type properties and elementary embeddings shows that the first cardinal $\nu \leq \mu$ such that $(\kappa, \nu) \to (\mu, < \nu)$ is regular. We shall prove that such ν cannot be strongly inaccessible, whenever $\mu = \kappa$.

LEMMA 3.1. *Assume that $(\kappa, \nu) \to (\kappa, < \nu)$ and $\nu < \kappa$ is a limit cardinal. Let $\lambda \leq \kappa$ be the least cardinal such that $\lambda > \nu$ and the property $(\kappa, \lambda) \to (\kappa, < \lambda)$ fails. If $\nu^{\mathrm{cf}(\nu)} < \lambda$, then there exists $\rho < \nu$ such that $(\kappa, \nu) \to (\kappa, < \rho)$.*

PROOF. Set $A = \{\sigma < \nu: (\kappa, \sigma) \to (\kappa, < \sigma) \text{ fails}\}$ and $\eta = \sup A$. If $\eta < \nu$, then our statement is true for $\rho = \eta^+$. If this were false, there would be some strictly increasing sequence $\langle \sigma_\alpha: \alpha < \mathrm{cf}(\nu) \rangle$ of elements of A, cofinal in ν. For each $\alpha < \mathrm{cf}(\nu)$ we can find a partition $f_\alpha: [\kappa]^{<\omega} \to \sigma_\alpha$ such that $|f''_\alpha[X]^{<\omega}| = \sigma_\alpha$ for every $X \subseteq \kappa$ of size κ (a counterexample for $(\kappa, \sigma_\alpha) \to (\kappa, < \sigma_\alpha)$).

Put $B = \prod_{\alpha < \mathrm{cf}(\nu)} \sigma_\alpha$ and define $g: [\kappa]^{<\omega} \to B$ setting $g(s) = \langle f_\alpha(s): \alpha < \mathrm{cf}(\nu) \rangle$ for each finite subset $s \subseteq \kappa$. As $|B| = \nu^{\mathrm{cf}(\nu)} < \lambda$, the definition of λ supplies a set $X \subseteq \kappa$ such that $|X| = \kappa$ and $|g''[X]^{<\omega}| < \nu$. On the other hand, $|g''[X]^{<\omega}| \geq \sup_{\alpha < \mathrm{cf}(\nu)} |f''_\alpha[X]^{<\omega}| \geq \nu$ by definition of g. This contradiction establishes the Lemma. □

COROLLARY 3.2. *If $\nu < \kappa$ is a strong limit cardinal and $(\kappa, \nu) \to (\kappa, < \nu)$, then $(\kappa, \nu) \to (\kappa, < \rho)$ for some $\rho < \nu$. Thus the least cardinal $\lambda < \kappa$ such that $(\kappa, \lambda) \to (\kappa, < \lambda)$ cannot be strongly inaccessible.*

PROOF. Let $\lambda \leq \kappa$ be the least cardinal such that $\lambda > \nu$ and $(\kappa, \lambda) \to (\kappa, < \lambda)$ fails. Observe that λ is ν-Rowbottom. Since $2^{<\nu} = \nu$, the cardinal λ is strong limit by Corollary 2.2. Now $\nu^{\mathrm{cf}(\nu)} = 2^\nu < \lambda$ and our claim follows from Lemma 3.1. □

QUESTION 3.3. Is the least $\lambda < \kappa$ such that $(\kappa, \lambda) \to (\kappa, < \lambda)$ is always a successor cardinal?

THEOREM 3.4. *Every strongly inaccessible Jónsson cardinal κ is ρ-Rowbottom for some $\rho < \kappa$.*

PROOF. Let $\lambda < \kappa$ be any regular cardinal such that $(\kappa, \lambda) \to (\kappa, < \lambda)$. The set $S = \{\nu < \kappa: \mathrm{cf}(\nu) = \lambda \text{ and } \nu \text{ is strong limit}\}$ is stationary in κ. Lemma 2 from [**8**] shows that $(\kappa, \nu) \to (\kappa, < \nu)$ for every $\nu \in S$. By Corollary 3.2, for each $\nu \in S$ we can choose some $\rho_\nu < \nu$ so that $(\kappa, \nu) \to (\kappa, < \rho_\nu)$. By Fodor's Theorem there exist some fixed $\rho < \kappa$ and a stationary subset $T \subseteq S$ such that $(\kappa, \nu) \to (\kappa, < \rho)$ for each $\nu \in T$. This means that κ is ρ-Rowbottom, since T is unbounded in κ. □

QUESTION 3.5. May we erase the word "strongly" from the above Theorem?

4. Jónsson models and successor cardinals. We showed in [**8**] that a successor cardinal κ^+ is not Jónsson, whenever κ is regular. Alternatively, κ^+ is not Jónsson under $2^\kappa = \kappa^+$ [**1**]. Shelah's method from [**6**] enables us to weaken this presumption.

We say that a regular cardinal μ is a possible scale for the sequence $\langle \kappa_i: i < \lambda \rangle$ of cardinals iff there exists a sequence $\langle f_\alpha: \alpha < \mu \rangle$ of functions on λ such that

(i) $f_\alpha \in \prod_{i<\lambda} \kappa_i$ for all $\alpha < \mu$,

(ii) $|\{i < \lambda: f_\alpha(i) \geq f_\beta(i)\}| < \lambda$ for all $\alpha < \beta < \mu$. (We shall write $f_\alpha < f_\beta$.)

(iii) For every $f \in \prod_{i<\lambda} \kappa_i$ there exists $\alpha < \mu$ such that $|\{i < \lambda: f(i) \leq f_\alpha(i)\}| = \lambda$ (compare [**6**]).

LEMMA 4.1 (SHELAH [**6**]). *Let* $j: M \to V_\gamma$ *and* $N = j''M$. *If* μ *is a possible sclae for the sequence* $\langle \kappa_i: i < \lambda \rangle \in N$ *of regular cardinals,* $\lambda + 1 \subseteq N$ *and* $j(\mu) = \mu$, *then* $|\{i < \lambda: j(\kappa_i) = \kappa_i\}| = \lambda$.

PROOF. Set $A = \{i < \lambda: |N \cap \kappa_i| < \kappa_i\}$ and assume to the contrary that $|\lambda \backslash A| < \lambda$. As N is the elementary substructure of V_γ, some sequence $\langle f_\alpha: \alpha < \mu \rangle \in N$ exemplifies that μ is a possible scale for $\langle \kappa_i: i < \lambda \rangle$. For each $i \in A$ the subset $B_i = \{f_\alpha(i): \alpha \in N \cap \mu\}$ of $N \cap \kappa_i$ has cardinality less than κ_i, so we may choose $\sup B_i < f(i) < \kappa_i$ by regularity of κ_i. Accepting $f(i) = 0$ for $i \in \lambda \backslash A$, we have $f_\alpha < f$ for every $\alpha \in N \cap \mu$. As the relation $<$ is transitive and the set $N \cap \mu$ is cofinal in μ, $f_\alpha < f$ for every $\alpha < \mu$, which is impossible. □

THEOREM 4.2. *If* $\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$, *then* κ^+ *is not Jónsson.*

PROOF. By our first remark in this item we may assume that $\lambda = \mathrm{cf}(\kappa) < \kappa$. Suppose that κ^+ is Jónsson and pick an elementary embedding $j: M \to V_{\kappa^+ + \omega}$ such that $j(\alpha) = \alpha$ for all $\alpha \leq \lambda$, $j(\nu) > \nu$ for some $\nu < \kappa$ and $j(\kappa^+) = \kappa^+$ (see the proof of Theorem 1 in [**8**]). Then there exists some strictly increasing sequence $\langle \kappa_i: i < \lambda \rangle \in j''M$ of cardinals, cofinal in κ, with $\kappa_0 \geq \nu$.

Cantor's diagonalization method shows that every family of κ functions $f \in \prod_{i<\lambda} \kappa_i^{++}$ has an upper bound in the relation $<$. Since $|\prod_{i<\lambda} \kappa_i^{++}| = \kappa^\lambda = \kappa^+$, κ^+ is the only possible scale for $\langle \kappa_i^{++}: i < \lambda \rangle$. Now, by Lemma 4.1, the set $A = \{i < \lambda: j(\kappa_i^{++}) = \kappa_i^{++}\}$ is unbounded in λ. But each κ_i^{++}, where $i \in A$, is Jónsson, contradicting [**8**]. □

COROLLARY 4.3 (SHELAH [**6**]). *If* $2^{\aleph_\alpha} = \aleph_{\alpha+\gamma+1}$ *and* $\mathrm{cf}(\gamma) < \aleph_{\alpha+1}$, *then* $2^{\aleph_\alpha}$ *cannot be Jónsson.* □

With a slight modification, a similar argument can be used for

LEMMA 4.4. *If* κ^+ *is Jónsson,* $\lambda = \mathrm{cf}(\kappa) > \omega$ *and* $\rho^\lambda < \kappa$ *for* $\rho < \kappa$, *then the set* $\{\rho < \kappa: \rho^+$ *is Jónsson*$\}$ *contains some closed unbounded subset of* κ.

PROOF. Let $j: M \to V_{\kappa^+ + \omega}$, $j(\alpha) = \alpha$ for all $\alpha \leq \lambda$, $j(\nu) > \nu$ for some $\lambda < \nu < \kappa$ and $j(\kappa^+) = \kappa^+$. Choose a strictly increasing continuous sequence $\langle \rho_i: i < \lambda \rangle \in j''M$ of cardinals, cofinal in κ, with $\rho_0 \geq \nu$. Suppose by way of contradiction that the set $S = \{i < \lambda: \rho_i^+ \text{ is not Jónsson}\}$ is stationary in λ.

Set $\kappa_i = \rho_i^+$ for $i \in S$ and $\kappa_i = \rho_i^{++}$ for $i \in \lambda \backslash S$. Thus no κ_i is Jónsson. Since S is stationary, by an analogue of Lemma 8.5 stated in [**3**], every family of almost disjoint functions $f \in \prod_{i<\lambda} \kappa_i$ has at most κ^+ elements. But every family of κ functions $f \in \prod_{i<\lambda} \kappa_i$ has an upper bound in the relation $<$. Thus κ^+ is the only possible scale for $\langle \kappa_i: i < \lambda \rangle$. Now each element of the set $\{\kappa_i: i < \lambda$ and $j(\kappa_i) = \kappa_i\} \neq \varnothing$ is Jónsson, contrary to our choice. □

REMARK 4.5. We recall another result of Shelah from [**6**] which can be formulated as follows: If $\rho^{\mathrm{cf}(\kappa)} < \kappa$ for all $\rho < \kappa$ and κ^+ is Jónsson, then the set $\{\lambda < \kappa: \lambda \text{ is a regular Jónsson cardinal}\}$ is unbounded in κ. Hence, if λ is arbitrary and $\lambda^\omega = \aleph_\alpha$, then $\aleph_{\alpha+\omega+1}$ cannot be Jónsson. □

We can also leave out one assumption in Shelah's result [**6**].

LEMMA 4.6. *If* $(\lambda^+)^\omega = \lambda^+$ *for all singular cardinals* λ, *then no successor cardinal is Jónsson.*

PROOF. Suppose to the contrary that κ^+ is the least successor cardinal which is Jónsson. There are now two cases:

Case I: $\mathrm{cf}(\kappa) = \omega$. Then $\kappa^{\mathrm{cf}(\kappa)} \leq (\kappa^+)^\omega = \kappa^+$ and a contradiction follows from Theorem 4.2.

Case II: $\kappa > \mathrm{cf}(\kappa) > \omega$. First, collapse $\mathrm{cf}(\kappa)$ onto ω_1 using the notion of forcing $P = \{p: p$ is a function with $\mathrm{dom}(p) \in \omega_1$ and $\mathrm{ran}(p) \subseteq \mathrm{cf}(\kappa)\}$ ordered by inclusion.

Let G be any generic filter on P. Since $|P| = \mathrm{cf}(\kappa)^\omega < \kappa$, it follows from [**8**] that κ^+ remains Jónsson in the forcing extension $V[G]$ of a ground model V. Clearly, since P is ω_1-closed, ω_1 is preserved and $\mathrm{cf}(\kappa) = \omega_1$ in $V[G]$. Moreover, $|P|$ is collapsed onto ω_1 and every cardinal $\lambda > |P|$ in V remains a cardinal in $V[G]$. The equality $(\lambda^+)^\omega = \lambda^+$ is also true in $V[G]$ for every singular cardinal λ.

From now on work in $V[G]$. Let $j: M \to V_\gamma$ witness that κ^+ is Jónsson. Pick some strictly increasing continuous sequence $\langle \kappa_i : i < \omega_1 \rangle \in j''M$ of cardinals with cofinality ω, cofinal in κ. Since $|\prod_{i<1} \kappa_i^+| \leq (\kappa_1^+)^\omega = \kappa_1^+$ for all $1 < \omega_1$, by Claim 13 from [**6**] the successor κ^+ is a possible scale for $\langle \kappa_i^+ : i < \omega_1 \rangle$. It follows from Lemma 4.1 that the set $A = \{i < \omega_1 : j(\kappa_i^+) = \kappa_i^+\}$ is unbounded in ω_1. But if $i \in A$ and κ_i^+ is greater than the first ordinal moved by j, then κ_i^+ is Jónsson, which is a contradiction because $\mathrm{cf}(\kappa_i) = \omega$ and $(\kappa_i)^\omega = \kappa_i^+$. □

The same proof shows

LEMMA 4.7. *Assume the Singular Cardinals Hypothesis. Then $2^\omega < \kappa$ implies that κ^+ is not Jónsson.* □

LEMMA 4.8. *No successor cardinal above a compact cardinal is Jónsson.*

PROOF. Solovay showed that the singular cardinals hypothesis holds above the least compact cardinal (see [**3**, p. 405]). Now proceed as in the proof of Lemma 4.6. □

REFERENCES

1. P. Erdös and A. Hajnal, *On a problem of B. Jónsson*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **14** (1966), 19–23.
2. F. Galvin and A. Hajnal, *Inequalities for cardinal powers*, Ann. of Math. (2) **101** (1975), 491–498.
3. T. Jech, *Set theory*, Academic Press, New York, 1978.
4. M. Magidor, *Chang's conjecture and powers of singular cardinals*, J. Symbolic Logic **42** (1977), 272–276.
5. S. Shelah, *A note on cardinal exponentiation*, J. Symbolic Logic **45** (1980), 56–66.
6. ——, *Jónsson algebras in successor cardinals*, Israel J. Math. **30** (1978), 57–64.
7. J. Tryba, *A few remarks on Rowbottom cardinals*, Israel J. Math. **40** (1981), 193–196.
8. ——, *On Jónsson cardinals with uncountable cofinality*, Israel J. Math. **49** (1984), 315–324.

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