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TARSKI'S PROBLEM FOR SOLVABLE GROUPS

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ABSTRACT. In this paper, we show that the free solvable groups (as well as the free nilpotent groups) of finite rank have different elementary theories (i.e., they do not satisfy the same first order sentences of group theory). This result is obtained using a result in group theory (probably due to Malcev and following immediately from a theorem of Auslander and Lyndon) that, for a free nontrivial solvable group, the last nontrivial group in its derived series is is own centralizer.

Introduction. The question of the elementary equivalence of (absolutely) free groups was posed by Tarski [**8, 9**]. Vaught [**10**] has shown that any two free groups of infinite rank are elementarily equivalent, and his proof easily extends to the free groups of infinite rank in any variety. The problem is still open for finitely generated free groups. However, it is known that they all share the same positive theory (Merzlyakov [**5**], Sacerdote [**6**]), and, as an immediate consequence of Sacerdote's work, the same $\forall$ theory. On the other hand, the sentences which distinguish the finitely generated free abelian groups are positive $\forall$, and, as shown in the third section, the same sentences can be used to distinguish between the finitely generated free nilpotent groups.

The main result of this paper is that the finitely generated free solvable groups have different positive $\forall\exists\forall$ theories (Theorem 2).

Preliminaries. The first order language L of group theory has individual variables $v_1, v_2, \ldots$ which range over group elements, the individual constant 1 for the identity element, the symbols $\cdot$ and $^{-1}$ for the group operations, the equality symbol $=$, and the logical symbols $\wedge$, $\vee$, $\forall$ and $\exists$. Well-formed formulae (and sentences) are defined in the usual way (see, e.g., [**2**]). All atomic formulae may be assumed to be of the form $W = 1$, where W is a word in the individual variables (i.e., a product of the variables and their inverses). Every formula is logically equivalent to a formula in prenex normal form $Q_1V_1 \cdots Q_mV_mM$, where M is a boolean combination of atomic formulae, each Q_i is either $\forall$ or $\exists$, and each V_i is a sequence of individual variables. M is called the matrix of the formula while the part of the formula to the left of M is called the prefix of the formula. A sentence of L is a formula in which each variable occurring in the matrix also occurs in the prefix; variables in the

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matrix not occurring in the prefix are called free variables. A formula of L will be called $\forall_n$ if it is logically equivalent to a formula whose Q_1 is $\forall$, whose Q_i alternate between $\forall$ and $\exists$, and whose $m \leqslant n$. In this paper we are interested in $n \leqslant 3$. A formula of L is called positive if it is logically equivalent to a formula whose matrix does not involve the negation symbol $\sim$. The (positive, $\forall_n$) theory of a group G is the set of all (positive, $\forall_n$) sentences of L true in G.

Two groups G and H are *elementarily equivalent*, written $G \equiv H$, if they have precisely the same true sentences in L (and hence the same theories in L). For any unexplained model-theoretic terminology or results used in this paper see [**2**]. For any unexplained group-theoretic terminology or results see [**3** or **4**].

Nilpotent groups. It is well known that if two groups are elementarily equivalent, then so are their quotient groups by normal subgroups definable by the same formula in each (a subgroup S is definable in a group G if there is some formula $F(v)$ with one free variable v in L such that the elements of G which make $F(v)$ true are precisely the elements of S). One may use this remark to extend the result that two free abelian groups of different finite ranks are not elementarily equivalent to free nilpotent groups. Indeed, in a free nilpotent group $G \neq 1$ of class n, the centre of G is the nth term G_n of the lower central series of G (see, e.g., problem 5.7.5, [**4**]). Hence G_n is definable in G. But G/G_n is a free nilpotent group of the same rank as G and has class $n-1$. Since a free abelian group is a free nilpotent group of class 1, by induction on the class n one shows that two free nilpotent groups of different finite ranks are not elementarily equivalent. However, the distinguishing sentences we obtain in this way are in $\forall_3$ (see Szmielew [**7**]). We show now that using the commutator calculus for free groups, one can obtain distinguishing sentences in $\forall_2$.

For any group G the subgroups G_n of the lower central series of G are defined recursively by $G_1 = G$ and $G_{n+1} = [G_n, G]$, where $[A, B]$ means the subgroup generated by the commutators $a^{-1}b^{-1}ab$ for all $a \in A$ and $b \in B$. G is said to be nilpotent of class c if c is the least positive integer for which $G_{c+1} = 1$. The free nilpotent group of class c and rank r is given by F/F_{c+1}, where F is the (absolute) free group of rank r generated by $a_1, \dots, a_r$. For each $i \geqslant 1$ let $a_{i,1}, \dots, a_{i,w(r,i)}$ denote the basic commutators of weight i in $a_1, \dots, a_r$ in their usual order (for a definition of basic commutators and their ordering, see, e.g., [**4**]). The basic commutators of weight i are free generators for the free abelian group F_i/F_{i+1} (see, e.g., [**4**]).

LEMMA 1. *For each $j \geqslant 1$, any element f of F may be written in the form*

$$f = \left(\prod_{i=1}^{j} \prod_{k=1}^{w(r,i)} a_{i,k}^{\varepsilon_{i,k}} \right) \alpha \rho^2, \tag{1}$$

where $\alpha \in F_{j+1}$, $\rho \in F$, and each $\varepsilon_{i,k} = 0$ or 1. Moreover the sequence $\varepsilon_{i,k}$ is unique.

PROOF. For $j = 1$ the result is clear. Suppose that $j \geqslant 1$ and f has the form given by (1). Since the basic commutators of weight $j+1$ are free abelian generators for the free abelian group F_{j+1}/F_{j+2}, it follows that, modulo F_{j+2}, α can be written in the form

$$\left(\prod_{k=1}^{w(r,j+1)} a_{j+1,k}^{\varepsilon_{j+1,k}}\right)\rho_1^2,$$

where $\rho_1 \in F_{j+1}$. But F_{j+1}/F_{j+2} is in the centre of F/F_{j+2}, and so, modulo F_{j+2}, $\rho_1^2\rho^2 = (\rho_1\rho)^2$. Thus f has the required form (1). Moreover, the uniqueness of the exponents $\varepsilon_{i,k}$ is easily proved by induction, since the basic commutators of weight $j+1$ are free abelian generators for the free abelian group F_{j+1}/F_{j+2}. □

COROLLARY. *For each $n \geqslant 1$ and $c \geqslant 1$ there exists a positive $\forall$ sentence $s(n,c)$ such that a free nilpotent group of class c and rank r satisfies $s(n,c)$ iff $n > \tau(r,c)$, where $\tau(r,c) = w(r,1) + \cdots + w(r,c)$.*

PROOF. Clearly $\tau(r,c)$ is strictly monotonic in r. Now, the uniqueness of the exponents $\varepsilon_{i,k}$ in (1) guarantees that, modulo F_{j+1}, distinct products of basic commutators of weight $\leqslant c$ in their usual order with exponents 0 or 1, even when multiplied by squares, are still different. Hence, if $E(n,i)$ for fixed n and $i = 1$ to 2^n ranges over the distinct n-terms sequences of 0 and 1, and $P(n,i)$ is the monomial obtained by multiplying together $v_j^{\varepsilon_j}$ for $1 \leqslant j \leqslant n$, where ε_j is the jth term of $E(n,i)$, then in the free nilpotent group of rank r and class c the sentence

$$s(n,c) = \forall v_1 \cdots v_n v w \prod_{i<j}\left(P(n,i)^{-1}P(n,j) = v^{-2}w^2\right)$$

is false if $n \leqslant \tau(r,c)$, but true if $n > \tau(r,c)$. □

As a generalization of the work of Szmielew [7] for abelian groups, i.e., nilpotent groups of class 1, we have the following theorem.

THEOREM 1. *Any two free nilpotent groups of different finite ranks have different positive $\forall\exists$ theories.*

PROOF. If the nilpotent classes of the two groups are different, then a universal sentence indicating the vanishing of a repeated commutator distinguishes between the two groups. If they have the same nilpotent class, then the proof is immediate from the preceding corollary. □

Solvable groups. The nth derived group $G^{(n)}$ of a group G is defined recursively by $G^{(0)} = G$, $G^{(n+1)} = [G^{(n)}, G^{(n)}]$ for $n \geqslant 0$ (we usually denote $G^{(1)}$ by G'). Similarly we define a commutator $c^{(n)}$ of derived weight n recursively by $c^{(0)} = g$, where g is any element of G, and $c^{(n+1)} = [x,y]$, where x and y are any commutators of derived weight n. G is said to be solvable of derived length d, if d is the least nonnegative integer such that $G^{(d)} = 1$. Clearly G is solvable of derived length $\leqslant d$ iff each commutator of derived weight d in G is 1. Thus solvability of derived length $\leqslant d$ is equivalent to the truth of the sentence $\forall v_1 \cdots v_k W = 1$, where $k = 2^n$ and W is the commutator of derived weight n constructed in the natural way from the free generators $v_1, \ldots, v_k$ of a free group. The free solvable group of derived length d and rank r is given by $F/F^{(d)}$, where F is the free group of rank r.

LEMMA 2. *If G is a free solvable group of derived length $d \geqslant 1$, then $G^{(d-1)}$ is definable by a positive universal formula.*

PROOF. We defer the proof of Lemma 2 until the next section. □

LEMMA 3. *Let N be a normal subgroup of a group G, and suppose N is definable in G by a positive universal formula $\varphi(v)$. Then for any positive $\forall\exists\forall$ sentence s, we may construct a positive $\forall\exists\forall$ sentence $\underline{s}$, such that G/N satisfies s iff G satisfies $\underline{s}$.*

PROOF. Now if $W(v_1, \ldots, v_k) = 1$ is an atomic formula, then it is true in G/N under substitution of $g_1N, \ldots, g_kN$ for $v_1, \ldots, v_k$ iff $\varphi(W(v_1, \ldots, v_k))$ is true in G under the substitution of $g_1, \ldots, g_k$ for $v_1, \ldots, v_k$. Hence $\underline{s}$ may be obtained from s by replacing each atomic formula $W = 1$ in the matrix of s by $\varphi(W)$ and absorbing the universal quantifier of φ into the rightmost universal quantifier in the prefix of s. □

THEOREM 2. *Any two free solvable groups of different finite ranks have different positive $\forall\exists\forall$ theories.*

PROOF. The derived length of a group G is $\leqslant d$ iff G satisfies a positive universal sentence whose matrix is the formula stating that the "formal" commutator of derived weight d is 1. Hence we may assume that the derived lengths of the groups are the same. We use induction on the derived length d. Now $G/G^{(d-1)}$ is the free solvable group of derived length $d - 1$ and rank r if G is the free solvable group of derived length d and rank r. Also if d is 1, then G is abelian and its rank can be distinguished by $\forall\exists$ theory. Hence we may use Lemmas 2 and 3 to establish the theorem. □

Commutator centralizers in free solvable groups. Throughout this section F will denote a free group of rank $r \geqslant 1$ (possibly infinite) and G will denote the free solvable group of rank r and derived length $n + 1$ given by $F/F^{(n+1)}$.

THEOREM 3. *The centralizer of $G^{(n)}$ in the free solvable group G of rank r and derived length $n + 1$ is itself for all $n \geqslant 0$.*

PROOF. This follows immediately from Auslander and Lyndon [**1**].

PROOF OF LEMMA 2. Let $\varphi(v)$ be the positive universal formula given by

$$\forall v_1 \cdots v_q([W, v] = 1), \tag{2}$$

where $q = 2^n$ and W is the "formal" commutator of derived weight n constructed in the natural way from $v_1, \ldots, v_q$ (e.g., for $n = 2$, $W = [[v_1, v_2], [v_3, v_4]]$). Then by Theorem 3, $\varphi(v)$ defines $G^{(n)}$ in a free solvable group G of derived length $n + 1$. □

REFERENCES

1. M. Auslander and R. C. Lyndon, *Commutator subgroups of free groups*, Amer. J. Math. **77** (1955), 929–931.

2. C. C. Chang and H. J. Keisler, *Model theory*, Studies in Logic and the Foundations of Mathematics, no. 73, North-Holland, Amsterdam, 1973.

3. R. C. Lyndon and P. E. Schupp, *Combinatorial group theory*, Springer-Verlag, New York, 1977.

4. W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory*, rev. ed., Dover, New York, 1976.

5. Y. I. Merzlyakov, *Positive formulae of free groups*. Algebra i Logika Sem. **5** (1966), 25–42. (Russian)

6. G. S. Sacerdote, *Elementary properties of free algebras*, Trans. Amer. Math. Soc. **178** (1973), 127–138.

7. W. Szmielew, *Elementary properties of abelian groups*, Fund. Math. **41** (1955), 203.

8. A. Tarski, *Some notions on the borderline of algebra and metamathematics*, Proc. Internat. Congr. Math., no. 1, Cambridge, Mass., 1950, pp. 705–720.

9. A. Tarski and R. L. Vaught, *Arithmetical extensions of relational systems*, Compositio Math. **13** (1955), 81–102.

10. R. L. Vaught, *On the airthmetical equivalence of free algebras*, Bull. Amer. Math. Soc. **61** (1955), 173–174.

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