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# HOMOGENEOUS BOREL SETS

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ABSTRACT. Topological characterizations of all zero-dimensional homogeneous absolute Borel sets are obtained; it turns out that there are $\omega_1$ such spaces. We use results from game theory—particularly, about Wadge classes.

**1. Introduction and preliminaries.** *All spaces under discussion are separable and metrizable.* We will assume that the reader is familiar with the main facts about absolute Borel sets (see [**3** or **7**]). Notation follows [**7**].

In this paper we describe and characterize all homogeneous Borel sets in the Cantor set that are not in $\Delta_3^0$ (i.e., they are not both $F_{\sigma\delta}$ and $G_{\delta\sigma}$). Together with [**1**], where all homogeneous Borel sets in $2^\omega$ of class $\Delta_3^0$ were determined, this yields a complete topological classification of all zero-dimensional homogeneous absolute Borel sets. Roughly, using the inductive definition of the non-self-dual Borel Wadge classes as given by Louveau [**5**], we show that the Wadge class of a non-$\Delta_3^0$ homogeneous Borel set in $2^\omega$ is non-self-dual and reasonably closed (for definitions, see below); then we can apply a theorem of Steel [**8**] to get what we want.

Let $Z$ be any space. If $\Gamma \subset \mathscr{P}(Z)$, then $\check{\Gamma} = \{A \subset Z\colon Z \setminus A \in \Gamma\}$, and $\Delta(\Gamma) = \Gamma \cap \check{\Gamma}$. $\Gamma$ is called *self-dual* if $\Gamma = \check{\Gamma}$. Mostly, we work inside the Cantor set $2^\omega$, denoted by $X$. Let $Q_i = \{x \in X\colon \exists m \forall n \geqslant m\colon x_n = i\}$, for $i \in \{0,1\}$. Then $Q_0 \approx Q_1 \approx \mathbf{Q}$, the space of rationals. If $x \notin Q_0 \cup Q_1$, then $x$ consists of blocks of zeros separated by blocks of ones; define $\phi\colon X \setminus (Q_0 \cup Q_1) \to X$ by $\phi(x)_n = 0$ if the $n$th block of zeros in $x$ has even length, and $\phi(x)_n = 1$ otherwise. Note that $\phi$ is continuous.

1.1 DEFINITION (STEEL [**8**]). (a) $\Gamma \subset \mathscr{P}(X)$ *is a reasonably closed pointclass if* $\phi^{-1}[A] \cup Q_0 \in \Gamma$ *for each* $A \in \Gamma$, *and* $f^{-1}[A] \in \Gamma$ *for each* $A \in \Gamma$ *and each continuous* $f\colon X \to X$. (b) $A \subset X$ *is everywhere properly* $\Gamma$ *if for each open* $U \neq \varnothing$ *in* $X$ *we have* $U \cap A \in \Gamma \setminus \check{\Gamma}$.

1.2 THEOREM (STEEL [**8**]). *If* $\Gamma$ *is a reasonably closed pointclass of Borel sets, and* $A, B \subset X$ *are everywhere properly* $\Gamma$ *and either both meager or both comeager, then* $h[A] = B$ *for some autohomeomorphism* $h$ *of* $X$.

Now let $Z \in \{X, \omega^\omega\}$. If $A, B \subset Z$, define $A \leqslant_{\mathrm{w}} B$ if $A = f^{-1}[B]$ for some continuous $f\colon Z \to Z$. The *Wadge class of* $A$ is $[A] = \{B \subset Z\colon B \leqslant_{\mathrm{w}} A\}$, and $\Gamma \subset \mathscr{P}(Z)$ is a *Borel Wadge class* in $Z$ if $\Gamma = [A]$ for some Borel set $A$ in $Z$. Define

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the *Wadge ordering* $\leqslant$ on the Wadge classes by $\Gamma_1 \leqslant \Gamma_2$ if $\Gamma_1 \subset \Gamma_2$, and $\Gamma_1 < \Gamma_2$ if $\Gamma_1 \leqslant \Gamma_2$ and $\Gamma_1 \neq \Gamma_2$. Using game theory, it can be shown that if $\Gamma_1$, $\Gamma_2$ are Borel Wadge classes in $Z$, then $\Gamma_1 < \Gamma_2$, $\Gamma_1 \in \{\Gamma_2, \check{\Gamma}_2\}$, or $\Gamma_2 < \Gamma_1$. Furthermore, if $\Gamma_1 < \Gamma_2$, then also $\Gamma_1 < \check{\Gamma}_2$ (and hence $\check{\Gamma}_1 < \check{\Gamma}_2, \Gamma_2$). Thus, if we consider $\leqslant$ to be an ordering on pairs $\{\Gamma, \check{\Gamma}\}$ of Borel Wadge classes, then $\leqslant$ becomes a linear ordering, and, in fact, a well-ordering in type $< \omega_2$ (see Wadge [**10**]; for some proofs, see [7]).

By van Wesep [**9**] the pattern of dual and non-self-dual Borel Wadge classes in the Wadge ordering on $\omega^\omega$ is as follows: The first element is $\{\{\varnothing\}, \{\omega^\omega\}\}$; a successor is self-dual if and only if its predecessor is not; at limit stages of cofinality $\omega$ stands a self-dual class; and at limit stages of cofinality $\omega_1$, a non-self-dual pair. Since we want to apply Theorem 1.2, we have to consider Borel Wadge classes in $X$ instead of $\omega^\omega$. In [**5**] Louveau has given construction principles "from below" for the Borel Wadge classes in $\omega^\omega$; but analyzing his results and proofs, it can be seen that the same inductive definition can be given for the Borel Wadge classes in $X$, with one exception: In the Borel Wadge ordering in $X$, the limit stages of cofinality $\omega$ are occupied by non-self-dual pairs (see Theorem 1.5(b)).

The following definitions and theorem are all due to Louveau for $\omega^\omega$ instead of $X$.

1.3 Definition. *Let* $\Gamma, \Gamma' \subset \mathscr{P}(X)$, *and let* $A \subset X$.

(a) $A \in D_\eta(\Sigma^0_\xi)$ *if there is an increasing sequence* $\langle A_\zeta\colon \zeta < \eta\rangle$ *of* $\Sigma^0_\xi$*-sets such that* $A = \bigcup_\zeta (A_\zeta \setminus \bigcup_{\beta<\zeta} A_\beta)$, *where* $\zeta$ *ranges over all even (odd) ordinals* $< \eta$ *if* $\eta$ *is odd (even).*

(b) $A \in \mathrm{Sep}(D_\eta(\Sigma^0_\xi), \Gamma)$ *if* $A = (A_1 \cap C) \cup (A_2 \setminus C)$ *for some* $C \in D_\eta(\Sigma^0_\xi)$, $A_1 \in \check{\Gamma}$, $A_2 \in \Gamma$.

(c) $A \in \mathrm{Bisep}(D_\eta(\Sigma^0_\xi), \Gamma, \Gamma')$ *if* $A = (A_1 \cap C_1) \cup (A_2 \cap C_2) \cup B \setminus (C_1 \cup C_2)$ *for some disjoint* $C_1, C_2 \in D_\eta(\Sigma^0_\xi)$, *and some* $A_1 \in \check{\Gamma}$, $A_2 \in \Gamma$, $B \in \Gamma'$.

(d) $A \in \mathrm{SU}(\Sigma^0_\xi, \Gamma)$ *if* $A = \bigcup_{n\in\omega}(A_n \cap C_n)$ *for some sequences* $\langle C_n\rangle_n$ *of pairwise disjoint* $\Sigma^0_\xi$*-sets,* $\langle A_n\rangle_n$ *of elements of* $\Gamma$. *The set* $\bigcup_{n\in\omega} C_n$ *is called the envelope of* $A$.

(e) $A \in \mathrm{SD}_\eta(\langle\Sigma^0_\xi, \mathrm{SU}(\Sigma^0_\xi, \Gamma)\rangle, \Gamma')$ *if* $A = \bigcup_{\zeta<\eta}(A_\zeta \setminus \bigcup_{\beta<\zeta} C_\beta) \cup B \setminus \bigcup_{\zeta<\eta} C_\zeta$ *for some increasing sequences* $\langle A_\zeta\colon \zeta < \eta\rangle$ *of elements of* $\mathrm{SU}(\Sigma^0_\xi, \Gamma)$, *and* $\langle C_\zeta\colon \zeta < \eta\rangle$ *of* $\Sigma^0_\xi$*-sets such that* $A_\zeta \subset C_\zeta \subset A_{\zeta+1}$ *and* $C_\zeta$ *is the envelop of* $A_\zeta$, *and some* $B \in \Gamma'$.

In (c), (e), we omit $\Gamma'$ if $\Gamma' = \{\varnothing\}$.

To simplify exposition, in writing, e.g., "$A \in \mathrm{Sep}(D_\eta(\Sigma^0_\xi), \Gamma)$, say $A = (A_1 \cap C) \cup (A_2 \setminus C)$", we always assume that the sets $A_1$, $A_2$, $C$ are chosen as in the above definition. Louveau now selects a certain subset $D$ of $\omega_1^\omega$, its elements being called *descriptions*, and for each $u \in D$, a non-self-dual Borel Wadge class $\Gamma_u$ is defined. Also, the *type* $t(u) \in \{0,1,2,3\}$ of a description $u$ is defined, and with each $u \in D$ of type 1, an element $\bar{u} \in D$ is associated, everything according to the following definition (where sometimes $v \in \omega_1^\omega$ is considered as a pair $\langle v_0, v_1\rangle$ or a sequence $\langle v_n\colon n \in \omega\rangle$ of elements of $\omega_1^\omega$; $\mathbf{0} \in \omega_1^\omega$ has all coordinates 0):

1.4 Definition. (a) $\mathbf{0} \in D$, $\Gamma_{\mathbf{0}} = \{\varnothing\}$, $t(\mathbf{0}) = 0$.

(b) *If* $u = \xi^\frown 1^\frown \eta^\frown \mathbf{0}$, *where* $\xi \geqslant 1$, $\eta \geqslant 1$, *then* $u \in D$, $\Gamma_u = D_\eta(\Sigma^0_\xi)$. *If* $\eta$ *is limit, then* $t(u) = 2$; *if* $\eta = \eta_0 + 1$, *then* $t(u) = 1$, *and* $\bar{u} = \mathbf{0}$ *if* $\eta_0 = \mathbf{0}$, $\bar{u} = \xi^\frown 1^\frown \eta_0^\frown \mathbf{0}$, *otherwise.*

(c) *If* $u = \xi^\frown 2^\frown \eta^\frown u^*$, *where* $\xi \geqslant 1$, $\eta \geqslant 1$, $u^* \in D$, $u^*(0) > \xi$, *then* $u \in D$, $\Gamma_u = \mathrm{Sep}(D_\eta(\Sigma^0_\xi), \Gamma_{u^*})$, *and* $t(u) = 3$.

(d) *If* $u = \xi^\frown 3^\frown \eta^\frown \langle u_0, u_1\rangle$, *where* $\xi \geqslant 1$, $\eta \geqslant 1$, $u_0$, $u_1 \in D$, $u_0(0) > \xi$, $u_1(0) \geqslant \xi$ *or* $u_1 = \mathbf{0}$, *and* $\Gamma_{u_1} < \Gamma_{u_0}$, *then* $u \in D$, $\Gamma_u = \mathrm{Bisep}(D_\eta(\Sigma^0_\xi), \Gamma_{u_0}, \Gamma_{u_1})$. *If* $u_1 = \mathbf{0}$ *and* $\eta = \eta_0 + 1$, *then* $t(u) = 1$, *and* $\bar{u} = u_0$ *if* $\eta_0 = 0$, $\bar{u} = \xi^\frown 2^\frown \eta_0^\frown u_0$, *otherwise. If* $u_1 = \mathbf{0}$ *and* $\eta$ *is limit, then* $t(u) = 2$. *If* $u_1(0) > \xi$, *then* $t(u) = 3$. *If* $u_1(0) = \xi$, *then* $t(u) = t(u_1)$, *and* $\bar{u} = \xi^\frown 3^\frown \eta^\frown \langle u_0, \bar{u}_1\rangle$ *if* $t(u_1) = 1$.

(e) *If* $u = \xi^\frown 4^\frown \langle u_n\colon n \in \omega\rangle$, *where* $\xi \geqslant 1$, *each* $u_n \in D$, $\Gamma_{u_n} < \Gamma_{u_{n+1}}$, $\langle u_n(0)\rangle_n$ *is nondecreasing and* $\sup u_n(0) > \xi$, *then* $u \in D$, $\Gamma_u = \mathrm{SU}(\Sigma^0_\xi, \bigcup_{n\in\omega}\Gamma_{u_n})$, $t(u) = 2$.

(f) *If* $u = \xi^\frown 5^\frown \eta^\frown \langle u_0, u_1\rangle$, *where* $\xi \geqslant 1$, $\eta \geqslant 2$, $u_0$, $u_1 \in D$, $u_0(0) = \xi$, $u_0(1) = 4$, $u_1(0) \geqslant \xi$ *or* $u_1 = \mathbf{0}$, *and* $\Gamma_{u_1} < \Gamma_{u_0}$, *then* $u \in D$, $\Gamma_u = \mathrm{SD}_\eta(\langle \Sigma^0_\xi, \Gamma_{u_0}\rangle, \Gamma_{u_1})$. *If* $u_1 = \mathbf{0}$ *then* $t(u) = 2$. *If* $u_1(0) > \xi$, *then* $t(u) = 3$. *If* $u_1(0) = \xi$, *then* $t(u) = t(u_1)$, *and* $\bar{u} = \xi^\frown 5^\frown \eta^\frown \langle u_0, \bar{u}_1\rangle$ *if* $t(u_1) = 1$.

1.5 Theorem. (a) *If* $t(u) = 1$ *and* $u(0) = 1$, *then* $\Gamma_{\bar{u}} < \Gamma_u$, *and* $\Delta(\Gamma_u)$ *is the unique Borel Wadge class* $\Gamma$ *such that* $\Gamma_{\bar{u}} < \Gamma < \Gamma_u$.

(b) *If* $t(u) = 2$ *and* $u(0) = 1$, *then* $\Delta(\Gamma_u) = \bigcup_{n\in\omega}\Gamma_{u_n}$ *for some strictly increasing sequence of described classes* $\langle \Gamma_{u_n}\rangle$ (*for* $\omega^\omega$, *Louveau has that* $\Delta(\Gamma_u)$ *is the unique Borel Wadge class* $\Gamma$ *such that* $\Gamma_{u_n} < \Gamma < \Gamma_u$ *for all* $n \in \omega$).

(c) *If* $u(0) > 1$ *or* $t(u) = 3$, *then* $\Delta(\Gamma_u) = \bigcup\{\Gamma_{u_\alpha}\colon \alpha \in \omega_1\}$ *for some strictly increasing sequence of described classes* $\langle \Gamma_{u_\alpha}\rangle_\alpha$.

From this theorem, it is easily deduced that $\{\Gamma_u\colon u \in D\} \cup \{\check{\Gamma}_u\colon u \in D\} \cup \{\Delta(\Gamma_u)\colon u \in D,\ t(u) = 1,\ u(0) = 1\}$ is the set of all Borel Wadge classes in $X$.

**2. Closure properties.** The statements in the following lemma were proved in Louveau [5] for Borel Wadge classes in $\omega^\omega$. However, the proofs work for $X$ as well.

2.1 Lemma. *Let* $u \in D$, $u(0) = \xi$.

(a) $\mathrm{SU}(\Sigma^0_\xi, \Gamma_u) = \Gamma_u$, *and if* $\eta < \xi$, *then* $\mathrm{SU}(\Sigma^0_\eta, \check{\Gamma}_u) = \check{\Gamma}_u$.

(b) $\Gamma_u$ *and* $\check{\Gamma}_u$ *are closed under union and intersection with a* $\Delta^0_\xi$-*set.*

(c) *If* $u(1) = 4$, *then* $\Gamma_u$ *is closed under union with a* $\Sigma^0_\xi$-*set.*

(d) *If* $t(u) = 3$, *then* $\Gamma_u$ *and* $\check{\Gamma}_u$ *are closed under union with a* $\Sigma^0_\xi$-*set and under intersection with a* $\Pi^0_\xi$-*set.*

(e) *If* $u = \xi^\frown 3^\frown \eta^\frown \langle u_0, u_1\rangle$, *and* $A \in \Gamma_u$, *then there exist* $C \in \Sigma^0_\xi$ *and* $B \in \Gamma_{u_1}$ *such that* $A = (A \cap C) \cup (B \setminus C)$, *and both* $A \cap C$ *and* $C \setminus A$ *are in* $\mathrm{Bisep}(D_\eta(\Sigma^0_\xi), \Gamma_{u_0})$.

(f) *If* $t(u) = 1$, *then* $\Gamma_u = \mathrm{Bisep}(\Sigma^0_\xi, \Gamma_{\bar{u}})$, *with* $\bar{u}$ *as defined in* 1.4.

(g) *If* $t(u) = 2$, *then* $\Gamma_u = \mathrm{SU}(\Sigma^0_\xi, \bigcup_{n\in\omega}\Gamma_{u_n})$ *for some strictly increasing sequence* $\langle \Gamma_{u_n}\rangle_n$ *of described classes with* $u_n(0) \geqslant \xi$ *for all* $n \in \omega$.

In this section we will prove more closure properties of some classes $\Gamma_u$, $\check{\Gamma}_u$ similar to (a)–(d) in the preceding lemma.

2.2 Lemma. *If* $\Delta^0_3 \subset \Gamma_u$ *and* $u(0) \geqslant 2$, *then* $\Gamma_u$ *is closed under intersection with a* $\Pi^0_2$-*set and under union with a* $\Sigma^0_2$-*set; hence so is* $\check{\Gamma}_u$.

PROOF. If not, there is a minimal class $\Gamma_u$ for which it fails. By 2.1(d) the lemma holds if $t(u) = 3$, and by 2.1(b) if $u(0) \geqslant 3$, so we have $u(0) = 2$ and $t(u) \in \{1, 2\}$.

*Case* 1. $t(u) = 1$. By 2.1(f), $\Gamma_u = \text{Bisep}(\Sigma_2^0, \Gamma_{\bar{u}})$. Since $\Delta_3^0 \subset \Gamma$, also $\Delta_3^0 \subset \Gamma_{\bar{u}}$ (otherwise $\Gamma_u = \Delta_3^0$, but $\Gamma_u$ is non-self-dual). In Definition 1.4 we see that in (b) we have $\Gamma_{\bar{u}} \not\supset \Delta_3^0$, so we must be in (d) or (f), whence $\bar{u}(0) \geqslant 2$. Since $\Gamma_{\bar{u}} \cup \check{\Gamma}_{\bar{u}} \subset \Gamma_u$, we have $\Gamma_{\bar{u}} < \Gamma_u$, so by minimality of $\Gamma_u$, $\Gamma_{\bar{u}}$ has the described closure properties. Let $A \in \Gamma_u$, say $A = (A_1 \cap C_1) \cup (A_2 \cap C_2)$. If $F \in \Pi_2^0$, then $A_1 \cap F \in \check{\Gamma}_{\bar{u}}$, $A_2 \cap F \in \Gamma_{\bar{u}}$, so $A \cap F = (A_1 \cap F \cap C_1) \cup (A_2 \cap F \cap C_2) \in \Gamma_u$. If $G \in \Sigma_2^0$, let $C_1^*$, $C_2^*$ reduce $C_1 \cup G$, $C_2 \cup G$. Then $A \cup G = ((A_1 \cup G) \cap C_1^*) \cup ((A_2 \cup G) \cap C_2^*) \in \Gamma_u$.

*Case* 2. $t(u) = 2$. By 2.1(g), $\Gamma_u = \text{SU}(\Sigma_2^0, \bigcup_{n \in \omega} \Gamma_{u_n})$. If each $\Gamma_{u_n} \subset \Delta_3^0$, then $\Gamma_u \subset \text{SU}(\Sigma_2^0, \Delta_3^0)$. Now if $v \in 3^\wedge 1^\wedge 1^\wedge \mathbf{0}$, then $\Gamma_v = \Sigma_3^0$, so by 2.1(a), we have $\text{SU}(\Sigma_2^0, \Delta_3^0) \subset \text{SU}(\Sigma_2^0, \Sigma_3^0) \cap \text{SU}(\Sigma_2^0, \Pi_3^0) = \Sigma_3^0 \cap \Pi_3^0 = \Delta_3^0$, so $\Gamma_u \subset \Delta_3^0$, a contradiction. Thus we conclude that $\Delta_3^0 \subset \Gamma_{u_n}$ for some $n$, and hence $\Delta_3^0 \subset \Gamma_{u_m}$ for all $m \geqslant n$. Since $u_m(0) \geqslant 2$ and $\Gamma_{u_m} < \Gamma_u$, each $\Gamma_{u_m}$ has the described closure properties; now proceed as in Case 1. □

2.3 LEMMA. *If $u(0) \geqslant 3$, or $u(0) = 2$ and $t(u) = 3$, then $\Gamma_u$ is closed under union with a $\Pi_2^0$-set.*

PROOF. If the lemma fails, there is a minimal $\Gamma_u$ for which it does. Since the lemma is true if $u(0) \geqslant 3$ by 2.1(b), we have $u(0) = 2$ and $t(u) = 3$. Let $F \in \Pi_2^0$.

*Case* 1. $u(1) = 2$, so $\Gamma_u = \text{Sep}(D_\eta(\Sigma_2^0), \Gamma_{u^*})$, $u^*(0) > 2$. If $A \in \Gamma_u$, say $A = (A_1 \cap C) \cup (A_2 \setminus C)$, then $A_1 \cup F \in \check{\Gamma}_{u^*}$, $A_2 \cup F \in \Gamma_{u^*}$ by 2.1(b), so $A \cup F = ((A_1 \cup F) \cap C) \cup (A_2 \cup F) \setminus C \in \Gamma_u$.

*Case* 2. $u(1) = 3$, so $\Gamma_u = \text{Bisep}(D_\eta(\Sigma_2^0), \Gamma_{u_0}, \Gamma_{u_1})$, $u_0(0) > 2$, and $u_1(0) > 2$ or $(u_1(0) = 2$ and $t(u_1) = 3)$. If $A \in \Gamma_u$, say

$$A = (A_1 \cap C_1) \cup (A_2 \cap C_2) \cup B \setminus (C_1 \cup C_2),$$

then $A_1 \cup F \in \check{\Gamma}_{u_0}$, $A_2 \cup \text{F} \in \Gamma_{u_0}$ by 2.1(b), and $B \cup F \in \Gamma_{u_1}$ by 2.1(b) if $u_1(0) > 2$, and by minimality of $\Gamma_u$ if $u_1(0) = 2$. So

$$A \cup F = ((A_1 \cup F) \cap C_1) \cup ((A_2 \cup F) \cap C_2) \cup (B \cup F) \setminus (C_1 \cup C_2) \in \Gamma_u.$$

*Case* 3. $u(1) = 5$, so $\Gamma_u = \text{SD}_\eta(\langle \Sigma_2^0, \Gamma_{u_0} \rangle, \Gamma_{u_1})$, $u_0(0) = 2$, $u_0(1) = 4$, and $u_1(0) > 2$ or $(u_1(0) = 2$ and $t(u_1) = 3)$. Let $A \in \Gamma_u$, say

$$A = \bigcup_{\zeta < \eta} \left( A_\zeta \setminus \bigcup_{\beta < \zeta} C_\beta \right) \cup B \setminus \bigcup_{\zeta < \eta} C_\zeta.$$

Since $X \setminus F \in \Sigma_2^0$, and $u_0(0) = 2$, by 2.1(a) we have $A_\zeta \cap (X \setminus F) \in \Gamma_{u_0}$, and it is easily verified that the envelop of $A_\zeta \cap (X \setminus F)$ is $C_\zeta \cap (X \setminus F)$. Also $B \cup F \in \Gamma_{u_1}$ as in Case 2. So

$$A \cup F = \bigcup_{\zeta < \eta} \left( (A_\zeta \cap (X \setminus F)) \setminus \bigcup_{\beta < \zeta} (C_\beta \cap (X \setminus F)) \right)$$
$$\cup (B \cup F) \setminus \bigcup_{\zeta < \eta} (C_\zeta \cap (X \setminus F)) \in \Gamma_u. \quad \square$$

2.4 COROLLARY. *If* $u(0) \geqslant 3$, *or* $u(0) = 2$ *and* $t(u) = 3$, *then* $\mathrm{SU}(\Sigma^0_2, \check{\Gamma}_u) = \check{\Gamma}_u$.

PROOF. If $A \in \mathrm{SU}(\Sigma^0_2, \check{\Gamma}_u)$, say $A = \bigcup_{n\in\omega}(A_n \cap C_n)$, then $X\setminus A = \bigcup_{n\in\omega}(C_n \cap X\setminus A_n) \cup X\setminus\bigcup_{n\in\omega}C_n$. Now $X\setminus A_n \in \Gamma_u$, so $\bigcup_{n\in\omega}(C_n \cap X\setminus A_n) \in \mathrm{SU}(\Sigma^0_2, \Gamma_u) = \Gamma_u$ by 2.1(a), and $X\setminus\bigcup_{n\in\omega}C_n \in \Pi^0_2$; thus, $X\setminus A \in \Gamma_u$ by 2.3, whence $A \in \check{\Gamma}_u$. □

**3. Existence of homogeneous Borel sets.** Some notation: If we apply the operations of 1.4 in a space $Z$, then we obtain classes $\Gamma_u(Z)$ (so $\Gamma_u = \Gamma_u(X)$). Inductively, it is easily shown that if $Z \subset X$, then $A \in \Gamma_u(Z)$ if and only if $A = B \cap Z$ for some $B \in \Gamma_u$, and similarly for $\check{\Gamma}_u$ (for the cases of 1.4(d), (e), use the reduction property). We write $Z_1 \approx Z_2$ if $Z_1$ is homeomorphic to $Z_2$, $h$: $Z_1 \approx Z_2$ if $h$ is a homeomorphism.

3.1 LEMMA. *If* $\Delta^0_3 \subset \Gamma_u$ *and* $u(0) \geqslant 2$, *and if* $B \subset X$, $A \in \Gamma_u$, $B \approx A$, *then* $B \in \Gamma_u$, *and similarly for* $\check{\Gamma}_u$.

PROOF. Let $f$: $A \approx B$. By Lavrentieff's theorem (see [**4** or **2**]), there exist $\Pi^0_2$-sets $G, H$ in $X$ with $A \subset G$, $B \subset H$, and a homeomorphism $\tilde{f}$: $G \to H$ extending $f$. Since $A \in \Gamma_u$, also $A \in \Gamma_u(G)$, so $B \in \Gamma_u(H)$, say $B = \tilde{B} \cap H$ with $\tilde{B} \in \Gamma_u$. Since $H \in \Pi^0_2$, by 2.2 we have $B \in \Gamma_u$. The proof for $\check{\Gamma}_u$ is analogous. □

3.2 DEFINITION. *If* $\Delta^0_3 \subset \Gamma_u$ *and* $u(0) \geqslant 2$, *then a zero-dimensional space* $Y$ *is everywhere properly* $\mathscr{P}_u$ (*resp.* $\check{\mathscr{P}}_u$) *if some copy of* $Y$ *in* $X$ *is everywhere properly* $\Gamma_u$ (*resp.* $\check{\Gamma}_u$).

Note that from 3.1 it follows that if $Y$ is everywhere properly $\mathscr{P}_u$ (resp. $\check{\mathscr{P}}_u$), then *each* dense embedding of $Y$ in $X$ is everywhere properly $\Gamma_u$ (resp. $\check{\Gamma}_u$), and that "everywhere properly $\mathscr{P}_u$" and "everywhere properly $\check{\mathscr{P}}_u$" are topological properties.

3.3 LEMMA. *If* $\Delta^0_3 \subset \Gamma_u$ *and* $u(0) \geqslant 2$, *let* $\mathscr{Y}^0_u$ ($\mathscr{Y}^1_u$), $\mathscr{X}^0_u$ ($\mathscr{X}^1_u$) *be the classes of all zero-dimensional spaces that are*, *respectively*, *everywhere properly* $\mathscr{P}_u$ *and first category* (*Baire*), *everywhere properly* $\check{\mathscr{P}}_u$ *and first category* (*Baire*). *Then up to homeomorphism*, *each class contains at most one space*, *and if it exists*, *this space is homogeneous*.

PROOF. To prove the first part of the lemma, it suffices to show that $\Gamma_u$ and $\check{\Gamma}_u$ are reasonably closed. For then if, e.g., $A, B \in \mathscr{Y}^0_u$, and $\tilde{A}, \tilde{B}$ are copies of $A, B$ in $X$ that are everywhere properly $\Gamma_u$, then $\tilde{A}$ and $\tilde{B}$ are meager, so we can apply Theorem 1.2; the other cases are similar. So let $\phi$ be as in §1, and put $P = X\setminus(Q_0 \cup Q_1)$. If $A \in \Gamma_u$, then clearly $\phi^{-1}[A] \in \Gamma_u(P)$. Hence for some $A' \in \Gamma_u$, we have $\phi^{-1}[A] = A' \cap P$. Since $P \in \Pi^0_2$, $\phi^{-1}[A] \in \Gamma_u$ by 2.2, and hence $\phi^{-1}[A] \cup Q_0 \in \Gamma_u$ by 2.2, since $Q_0 \in \Sigma^0_2$. The proof for $\check{\Gamma}_u$ is the same.

For the second part of the lemma, note that if $A$ is in one of the defined classes, then any nonempty open-and-closed subset of $A$ is in the same class since $u(0) \geqslant 2$ (use 2.1(a)), and hence it is homeomorphic to $A$; such a space is called *strongly homogeneous*, and it is not hard to show that any strongly homogeneous zero-dimensional space is homogeneous (see e.g. [**6**]). □

Thus, if $Y$ is in one of the above classes, then $Y$ is a homogeneous space that is topologically characterized by the properties describing the class.

We now determine which of the classes are nonempty.

3.4 LEMMA. *If* $\Delta_3^0 \subset \Gamma_u$ *and* $u(0) \geqslant 2$, *then* $\mathscr{Y}_u^0$ *and* $\mathscr{X}_u^1$ *are nonempty.*

PROOF. Let $Z \in \check{\Gamma}_u \setminus \Gamma_u$, and let $O = \bigcup\{U\colon U \text{ open in } Z, U \in \Gamma_u\}$. Then for some $U_n \in \Gamma_u$ with $U_n$ open in $Z$, we have $O = \bigcup_{n\in\omega} U_n$. Let $\tilde{U}_n$ be open in $X$ with $\tilde{U}_n \cap Z = U_n$, and let $\langle V_n \rangle_n$ reduce $\langle \tilde{U}_n \rangle_n$. Then $V_n \cap Z = V_n \cap \tilde{U}_n \cap Z = V_n \cap U_n$, $U_n \in \Gamma_u$, $u(0) \geqslant 2$, $V_n \in \Sigma_1^0$, so by 2.1(b), $V_n \cap Z \in \Gamma_u$. So $O = \bigcup_{n\in\omega}(V_n \cap Z) \in \mathrm{SU}(\Sigma_1^0, \Gamma_u) = \Gamma_u$ by 2.1(a). Put $\tilde{Z} = Z \setminus O$; then $\tilde{Z} \neq \varnothing$ since $Z \notin \Gamma_u$. Since $\tilde{Z} = Z \setminus \bigcup_{n\in\omega} V_n$, and $X \setminus \bigcup_{n\in\omega} V_n \in \Pi_1^0 \subset \Delta_2^0$, we have $\tilde{Z} \in \check{\Gamma}_u$ by 2.1(b). We claim that no nonempty open subset $U$ of $\tilde{Z}$ is in $\Gamma_u$. Indeed, if $U \in \Gamma_u$, choose $\tilde{U}$ open in $X$ with $\tilde{U} \cap \tilde{Z} = U$; then

$$\tilde{U} \cap Z = \left(\left(X \setminus \bigcup_{n\in\omega} V_n\right) \cap (\tilde{U} \cap \tilde{Z})\right) \cup \bigcup_{n\in\omega}((\tilde{U} \cap V_n) \cap (V_n \cap Z))$$

$$\in \mathrm{SU}(\Sigma_2^0, \Gamma_u) = \Gamma_u$$

by 2.1(a), so $\tilde{U} \cap Z \subset O$, contradicting $\varnothing \neq U \subset (\tilde{U} \cap Z) \setminus O$. Now let $Z'$ be a densely embedded copy of $\tilde{Z}$ in $X$ (which exists since $\tilde{Z}$ contains no isolated points), and put $Y = X \setminus Z'$. Also, let $Q$ be a countable dense subset of $X$, and put $Y_u^0 = Q \times Y \subset X \times X$. We identify $X \times X$ with $X$, and claim that $Y_u^0 \in \mathscr{Y}_u^0$. First note that, by 3.1, $Z' \in \check{\Gamma}_u$, hence $Y \in \Gamma_u$ and $\{q\} \times Y \in \Gamma_u$, so

$$Q \times Y = \bigcup_{q\in Q} (\{q\} \times Y \cap \{q\} \times X) \in \mathrm{SU}(\Sigma_2^0, \Gamma_u) = \Gamma_u.$$

Now if $V$ is a nonempty open subset of $X \times X$, then $V \cap Y_u^0 \neq \varnothing$, say $(q, x) \in V \cap Y_u^0$. Then $U = (\{q\} \times Y) \cap V$ is a nonempty open subset of $\{q\} \times Y$, and also it is closed in $V \cap Y_u^0$. So if $V \cap Y_u^0$ were in $\check{\Gamma}_u$, then also $U \in \check{\Gamma}_u$ by 2.1(b); but then $Y$ contains a nonempty open subset $U'$ with $U' \in \check{\Gamma}_u$, say $U' = \tilde{U} \cap Y$ with $\tilde{U}$ open in $X$. Then $\tilde{U} \cap (X \setminus U) = \tilde{U} \cap Z'$ is a nonempty open subset of $Z'$ which is in $\Gamma_u$ since $\tilde{U} \in \Sigma_1^0$, $X \setminus U \in \Gamma_u$, $u(0) \geqslant 2$, a clear contradiction. Thus, $Y_u^0$ is everywhere properly $\Gamma_u$, and obviously it is first category; so $Y_u^0 \in \mathscr{Y}_u^0$. Arguing as above, it is easily seen that $(X \times X) \setminus Y_u^0 \in \mathscr{X}_u^1$. □

If we try to prove that $\mathscr{X}_u^0$ and $\mathscr{Y}_u^1$ are nonempty by replacing $\Gamma_u$ in the above proof by $\check{\Gamma}_u$, then we see that we need $\mathrm{SU}(\Sigma_2^0, \check{\Gamma}_u) = \check{\Gamma}_u$; as we shall see in Lemma 3.6, this is not always the case. However, from 2.4 we see that the following holds:

3.5 LEMMA. *If* $\Delta_3^0 \subset \Gamma_u$, *and* $u(0) \geqslant 3$ *or* $(u(0) = 2$ *and* $t(u) = 3)$, *then* $\mathscr{X}_u^0$ *and* $\mathscr{Y}_u^1$ *are nonempty.* □

In fact, if $Z_u^1$ and $Q$ are as in the proof of 3.4, then $Z_u^0 = Q \times Z_u^1 \in \mathscr{X}_u^0$, and $X^3 \setminus Z_u^0 \in \mathscr{Y}_u^1$.

3.6 LEMMA. *If* $\Delta_3^0 \subset \Gamma_u$, $u(0) = 2$, *and* $t(u) \in \{1, 2\}$, *then* $\mathscr{Y}_u^1 = \varnothing = \mathscr{X}_u^0$.

PROOF. If $Z \in \mathscr{X}_u^0$ is densely embedded in $X$, then $X \setminus Z \in \mathscr{Y}_u^1$ (argue as in the proof of 3.4), so it suffices to show that $\mathscr{Y}_u^1 = \varnothing$. We will prove that if $A \subset X$ is everywhere properly $\Gamma_u$, then $A$ is first category. First take $t(u) = 2$. By 2.1(g), $\Gamma_u = \mathrm{SU}(\Sigma_2^0, \bigcup_{n \in \omega} \Gamma_{u_n})$ so we can write $A = \bigcup_{n \in \omega}(A_n \cap C_n)$. Let $C_n = \bigcup_{m \in \omega} C_m^n$, with $C_m^n \in \Pi_1^0$; then if $A_n \in \Gamma_{u_k}$, also $C_m^n \cap A = C_m^n \cap A_n \in \Gamma_{u_k}$ since $u_k(0) \geqslant 2$, $C_m^n \in \Delta_2^0$, using 2.1(b). If $U \neq \varnothing$ is open in $A$, say $U = \tilde{U} \cap A$ with $\tilde{U}$ open in $X$, and if $U \subset C_m^n \cap A_n$, then $U = \tilde{U} \cap C_m^n \cap A_n \in \Gamma_{u_k} \subset \check{\Gamma}_u$ since $\Gamma_{u_k} < \Gamma_u$, a contradiction. So $C_m^n \cap A$ is closed and nowhere dense in $A$, whence

$$A = \bigcup_{n, m \in \omega} (C_m^n \cap A)$$

is first category. If $t(u) = 1$, note that since $\Delta_3^0 \subset \Gamma_u$, we have $\bar{u} \neq \mathbf{0}$ whence $\bar{u}(0) \geqslant 2$. Since $\Gamma_u = \mathrm{Bisep}(\Sigma_2^0, \Gamma_{\bar{u}})$ by 2.1(f), we can argue as above. □

**4. The Wadge class of a homogeneous Borel set.** In this section we show that if $Y$ is a homogeneous zero-dimensional absolute Borel set, and $Y \notin \Delta_3^0$, then $Y \in \mathscr{Y}_u^0 \cup \mathscr{Y}_u^1 \cup \mathscr{X}_u^0 \cup \mathscr{X}_u^1$ for some $u \in D$, $u(0) \geqslant 2$, $\Delta_3^0 \subset \Gamma_u$.

4.1 LEMMA. *Let $Y$ be a homogeneous Borel set in $X$ with $Y \notin \Delta_3^0$. Let $\Gamma_u$ be the least described class such that $A \in \Gamma_u \cup \check{\Gamma}_u$ for some nonempty open subset $A$ of $Y$. Then $\Delta_3^0 \subset \Gamma_u$ and $u(0) \geqslant 2$.*

PROOF. If $\Delta_3^0 \not\subset \Gamma_u$, then $\Gamma_u \cup \check{\Gamma}_u \subset \Delta_3^0$ whence $A \in \Delta_3^0$. Let $x \in A$, and by homogeneity of $Y$, let $h_y\colon Y \approx Y$ be such that $h_y(x) = y$. Take a countable subcovering $\{h_{y_n}[A]\colon n \in \omega\}$ of the open covering $\{h_y[A]\colon y \in Y\}$ of $Y$, and let $U_n$ be open in $X$ such that $U_n \cap Y = h_{y_n}[A]$. If $\langle V_n \rangle_n$ reduces $\langle U_n \rangle_n$, then

$$V_n \cap Y = V_n \cap U_n \cap Y = V_n \cap h_{y_n}[A] \in \Delta_3^0,$$

so $Y = \bigcup_{n \in \omega}(V_n \cap Y) \in \mathrm{SU}(\Sigma_1^0, \Delta_3^0) = \Delta_3^0$ (see the proof of 2.2 Case 2), a contradiction. So $\Delta_3^0 \subset \Gamma_u$.

Now assume $u(0) = 1$.

*Case* 1. $t(u) = 2$. By 2.1(g), $\Gamma_u = \mathrm{SU}(\Sigma_1^0, \bigcup_{n \in \omega} \Gamma_{u_n})$. If $A \in \Gamma_u$, say $A = \bigcup_{n \in \omega}(A_n \cap C_n)$, then some $C_n \cap A_n = C_n \cap A$ is a nonempty open subset of $A$, hence of $Y$; but $A_n \in$ some $\Gamma_{u_k}$, $u_k(0) \geqslant 1$, so $C_n \cap A_n \in \Gamma_{u_k}$ by 2.1(a), contradicting minimality of $\Gamma_u$ since $\Gamma_{u_k} < \Gamma_u$. If $A \in \check{\Gamma}_u$, then $X \setminus A = \bigcup_{n \in \omega}(A_n \cap C_n)$, so $A = \bigcup_{n \in \omega}(C_n \cap X \setminus A_n) \cup X \setminus \bigcup_{n \in \omega} C_n$. Since $\check{\Gamma}_{u_k} < \Gamma_{u_{k+1}}$, each $X \setminus A_n \in \bigcup_k \Gamma_{u_k}$, so if some $C_n \cap X \setminus A_n \neq \varnothing$, we obtain a contradiction as above; thus, $A = X \setminus \bigcup_{n \in \omega} C_n \in \Pi_1^0 \subset \Delta_3^0$, which is impossible.

*Case* 2. $u = 1^\frown 1^\frown \eta + 1^\frown \mathbf{0}$. Then $\Gamma_u = D_{\eta+1}(\Sigma_1^0)$, so $A \in \Delta_3^0$.

*Case* 3. $u = 1^\frown 2^\frown \eta^\frown u^*$. Then $\Gamma_u = \mathrm{Sep}(D_\eta(\Sigma_1^0), \Gamma_{u^*})$, $u^*(0) \geqslant 2$. If $A \in \Gamma_u$, then $A = (A_1 \cap C) \cup (A_2 \setminus C)$, $C \in D_\eta(\Sigma_1^0)$. Let $C = \bigcup_\zeta (C_\zeta \setminus \bigcup_{\beta < \zeta} C_\beta)$ as in Definition 1.3(a).

(i) If $C_\zeta \cap A = \varnothing$ for all $\zeta < \eta$, then $A = A_2 \setminus \bigcup_{\zeta < \eta} C_\zeta$. Since $A_2 \in \Gamma_{u^*}$ and $X \setminus \bigcup_{\zeta < \eta} C_\zeta \in \Pi_1^0 \subset \Delta_2^0$, we have $A \in \Gamma_{u^*}$ by 2.1(b); but $\Gamma_{u^*} < \Gamma_u$, contradicting minimality of $\Gamma_u$.

(ii) Let $\alpha < \eta$ be minimal with $C_\alpha \cap A \neq \varnothing$. If $\alpha$ and $\eta$ are both even or both odd, then $C_\alpha \setminus \bigcup_{\beta<\alpha} C_\beta \subset X \setminus C$, so $C_\alpha \cap A = C_\alpha \cap A_2 \setminus C$. Since $C_\alpha \cap X \setminus C \in \Delta_2^0$, $C_\alpha \cap A \in \Gamma_{u^*}$ as above, and $C_\alpha \cap A$ is a nonempty open subset of $Y$. If $\alpha$ is even and $\eta$ is odd, or conversely, then $C_\alpha \setminus \bigcup_{\beta<\alpha} C_\beta \subset C$, so $C_\alpha \cap A = C_\alpha \cap C \cap A_1 \in \check{\Gamma}_{u^*}$, which again is impossible. If $A \in \check{\Gamma}_u$, then $X \setminus A = (A_1 \cap C) \cup (A_2 \setminus C)$, so $A = ((X \setminus A_1) \cap C) \cup (X \setminus A_2) \setminus C$. Put $\tilde{A}_1 = X \setminus A_1 \in \Gamma_{u^*}$, $\tilde{A}_2 = X \setminus A_2 \in \check{\Gamma}_{u^*}$, and argue as above.

*Case* 4. $u = 1^\frown 3^\frown 1^\frown \langle u_0, \mathbf{0} \rangle$, $u_0(0) \geqslant 2$. Then $\Gamma_u = \mathrm{Bisep}(\Sigma_1^0, \Gamma_{u_0})$ by 1.4(d) and 2.1(f). If $A \in \Gamma_u$, then $A = (A_1 \cap C_1) \cup (A_2 \cap C_2)$. Now $C_i \cap A = C_i \cap A_i$, and either $A_1 \cap C_1$ or $A_2 \cap C_2$ is nonempty. Since $C_i \in \Sigma_1^0 \subset \Delta_2^0$, we obtain from 2.1(b) that $A_1 \cap C_1 \in \check{\Gamma}_{u_0}$, $A_2 \cap C_2 \in \Gamma_{u_0}$, so we have a contradiction.

If $A \in \check{\Gamma}_u$, then $X \setminus A = (A_1 \cap C_1) \cup (A_2 \cap C_2)$, so $A = (C_1 \cap X \setminus A_1) \cup (C_2 \cap X \setminus A_2) \cup X \setminus (C_1 \cup C_2)$. Since $X \setminus A_1 \in \Gamma_{u_0}$ and $X \setminus A_2 \in \check{\Gamma}_{u_0}$, we must have $C_i \cap X \setminus A_i = \varnothing$ by the above argument, so $A = X \setminus (C_1 \cup C_2) \in \Pi_1^0 \subset \Delta_3^0$, another contradiction.

*Case* 5. $u = 1^\frown 3^\frown \eta + 1^\frown \langle u_0, \mathbf{0} \rangle$, $\eta \geqslant 1$, $u_0(0) \geqslant 2$. Then $\Gamma_u = \mathrm{Bisep}(\Sigma_1^0, \Gamma_{\bar{u}})$, where $\Gamma_{\bar{u}} = \mathrm{Sep}(D_\eta(\Sigma_1^0), \Gamma_{u_0})$. It suffices to show that $\Gamma_{\bar{u}}$ and $\check{\Gamma}_{\bar{u}}$ are closed under intersection with a $\Sigma_1^0$-set, for then we can copy the proof of Case 4, replacing $\Gamma_{u_0}$ by $\Gamma_{\bar{u}}$. For $\Gamma_{\bar{u}}$, this follows from 2.1(a); for $\check{\Gamma}_{\bar{u}}$, it is equivalent to $\Gamma_{\bar{u}}$ being closed under union with a $\Pi_1^0$-set, and this is proved exactly as 2.3 Case 1.

*Case* 6. $u = 1^\frown 3^\frown \eta^\frown \langle u_0, u_1 \rangle$, $u_0(0) \geqslant 2$, $u_1(0) \geqslant 1$. Then $\Gamma_u = \mathrm{Bisep}(D_\eta(\Sigma_1^0), \Gamma_{u_0}, \Gamma_{u_1})$. If $A \in \Gamma_u$, then by 2.1(e) we can write $A = (A \cap C) \cup B \setminus C$ for some $C \in \Sigma_1^0$, $B \in \Gamma_{u_1}$ such that $A \cap C \in \mathrm{Bisep}(D_\eta(\Sigma_1^0), \Gamma_{u_0})$. Since $A \cap C$ is open in $A$, and since it is easily checked that $\mathrm{Bisep}(D_\eta(\Sigma_1^0), \Gamma_{u_0}) < \Gamma_u$ (use that $X \in \Gamma_{u_1}$ since $u_1(0) \geqslant 1$), we must have $A \cap C = \varnothing$, whence $A = B \setminus C$. But $B \in \Gamma_{u_0}$, $X \setminus C \in \Delta_2^0$, $u_0(0) \geqslant 2$, so $B \setminus C \in \Gamma_{u_0}$ by 2.1(b), so $A \in \Gamma_{u_0} < \Gamma_u$, a contradiction. If $A \in \check{\Gamma}_u$, then again by 2.1(e), we have $X \setminus A = ((X \setminus A) \cap C) \cup B \setminus C$ for some $C \in \Sigma_1^0$, $B \in \Gamma_{u_1}$ with $C \setminus (X \setminus A) = C \cap A \in \mathrm{Bisep}(D_\eta(\Sigma_1^0), \Gamma_{u_0})$. Thus $A = (A \cap C) \cup (X \setminus B) \setminus C$, whence, as above, $A = (X \setminus B) \setminus C$. But $X \setminus B \in \check{\Gamma}_{u_1} < \Gamma_{u_0}$, so we again obtain a contradiction.

*Case* 7. $u = 1^\frown 5^\frown \eta^\frown \langle u_0, u_1 \rangle$, $u_0(0) = 1$, $u_0(1) = 4$, $u_1(0) \geqslant 1$. Then $\Gamma_u = \mathrm{SD}_\eta(\langle \Sigma_1^0, \Gamma_{u_0} \rangle, \Gamma_{u_1})$. If $A \in \Gamma_u$, then $A = \bigcup_{\zeta<\eta}(A_\zeta \setminus \bigcup_{\beta<\zeta} C_\beta) \cup B \setminus \bigcup_{\zeta<\eta} C_\zeta$. Put $C = \bigcup_{\zeta<\eta} C_\zeta$. Again, it is easily checked that $\mathrm{SD}_\eta(\langle \Sigma_1^0, \Gamma_{u_0} \rangle) < \Gamma_u$, so since $C \in \Sigma_1^0$ and $C \cap A = \bigcup_{\zeta<\eta}(A_\zeta \setminus \bigcup_{\beta<\zeta} C_\beta) \in \mathrm{SD}_\eta(\langle \Sigma_1^0, \Gamma_{u_0} \rangle)$, we have $C \cap A = \varnothing$, so $A = B \setminus C$. By 2.1(c), $\Gamma_{u_0}$ is closed under union with a $\Sigma_1^0$-set, so $\check{\Gamma}_{u_0}$ is closed under intersection with a $\Pi_1^0$-set. Since $B \in \Gamma_{u_1} \subset \check{\Gamma}_{u_0}$ and $X \setminus C \in \Pi_1^0$, we have $A \in \check{\Gamma}_{u_0} < \Gamma_u$. If $A \in \Gamma_u$, and $X \setminus A = \bigcup_{\zeta<\eta}(A_\zeta \setminus \bigcup_{\beta<\zeta} C_\beta) \cup B \setminus \bigcup_{\zeta<\eta} C_\zeta$, then use Lemma 2.1 to show that

$$A = \bigcup_{\zeta<\eta}\left(\left((C_\zeta \setminus A_\zeta) \cup \bigcup_{\beta<\zeta} C_\beta\right) \setminus \bigcup_{\beta<\zeta} C_\beta\right) \cup (X \setminus B) \setminus \bigcup_{\zeta<\eta} C_\zeta$$

$$\in \mathrm{SD}_\eta\left(\left\langle \Sigma_1^0, \Gamma_{u_0} \right\rangle, \check{\Gamma}_{u_1}\right),$$

and argue as above. □

4.2 LEMMA. *Let $Y$ be a homogeneous zero-dimensional absolute Borel set with $Y \notin \Delta_3^0$. Then for some $u \in D$ with $\Delta_3^0 \subset \Gamma_u$ and $u(0) \geqslant 2$, we have $Y \in \mathscr{Y}_u^0 \cup \mathscr{Y}_u^1 \cup \mathscr{Z}_u^0 \cup \mathscr{Z}_u^1$.*

PROOF. Embed $Y$ densely in $X$, and let $\Gamma$ be the least Borel Wadge class such that $A \in \Gamma \cup \check{\Gamma}$ for some nonempty open subset $A$ of $Y$. If $\Gamma$ is self-dual, then $\Gamma = \Delta(\Gamma_v)$ for some $v \in D$ with $t(v) = 1$, $v(0) = 1$ (see §1). But then $\Gamma_v$ is the least described class such that $B \in \Gamma_v \cup \check{\Gamma}_v$ for some nonempty open $B$ in $Y$, contradicting 4.1. So $\Gamma$ is non-self-dual, say $\Gamma = \Gamma_u$, and by 4.1, $\Delta_3^0 \subset \Gamma_u$ and $u(0) \geqslant 2$. Let $x \in A$, and for each $y \in Y$, let $h_y$: $Y \approx Y$ be such that $h_y(x) = y$. Let $\{h_{y_n}[A]$: $n \in \omega\}$ be a countable subcovering of the open covering $\{h_y[A]$: $y \in Y\}$ of $Y$, and let $U_n$ be open in $X$ such that $U_n \cap Y = h_{y_n}[A]$. If $\langle V_n \rangle_n$ reduces $\langle U_n \rangle_n$, then

$$V_n \cap Y = V_n \cap U_n \cap Y = V_n \cap h_{y_n}[A].$$

Now $h_{y_n}[A] \in \Gamma_u$ (resp. $\check{\Gamma}_u$) if $A \in \Gamma_u$ (resp. $\check{\Gamma}_u$) by 3.1. Since $V_n \in \Sigma_1^0$ and $u(0) \geqslant 2$, we have $V_n \cap Y \in \Gamma_u$ (resp. $\check{\Gamma}_u$) by 2.1(b). So $Y \in \mathrm{SU}(\Sigma_1^0, \Gamma_u) = \Gamma_u$ (resp. $Y \in \mathrm{SU}(\Sigma_1^0, \check{\Gamma}_u) = \check{\Gamma}_u$) by 2.1(a). A similar argument shows that if $B \neq \varnothing$ is open in $X$, and $B \cap Y$ were in $\check{\Gamma}_u$ (resp. $\Gamma_u$), then we would have $Y \in \check{\Gamma}_u$ (resp. $Y \in \Gamma_u$), since $B \cap Y \neq \varnothing$, so $Y \in \Delta(\Gamma_u)$. Hence by 1.5(c), $Y \in \Gamma_v$ for some $\Gamma_v < \Gamma_u$, contradicting minimality of $\Gamma_u$. Thus $Y$ is everywhere properly $\Gamma_u$ (resp. everywhere properly $\check{\Gamma}_u$), and since a homogeneous space is either first category or Baire, the result follows. □

REMARK. In the above lemma, we in fact have that $Y \in \mathscr{Y}_u^0 \cup \mathscr{Y}_u^1$ if $[Y] = \Gamma_u$, and $Y \in \mathscr{Y}_u^0 \cup \mathscr{Y}_u^1$ if $[Y] = \check{\Gamma}_u$.

We can now formulate our main theorem. Let $D_0 = \{u \in D\colon \Delta_3^0 \subset \Gamma, u(0) \geqslant 2\}$, and $D_1 = \{u \in D_0\colon u(0) \geqslant 3 \text{ or } t(u) = 3\}$. By 3.4 and 3.5, for each $u \in D_0$, there are elements $Y_u^0$, $Z_u^1$ in $\mathscr{Y}_u^0$, $\mathscr{Z}_u^1$, respectively, and for each $u \in D_1$, there are elements $Y_u^1$, $Z_u^0$ in $\mathscr{Y}_u^1$, $\mathscr{Z}_u^0$, respectively.

4.3 THEOREM. *Up to homeomorphism, $\{Y_u^0, Z_u^1\colon u \in D_0\} \cup \{Y_u^1, Z_u^0\colon u \in D_1\}$ consists precisely of all homogeneous zero-dimensional absolute Borel sets outside $\Delta_3^0$.*

PROOF. Apply 3.3, 3.6, and 4.2. □

Thus, from the remark following 4.2 and from the above theorem, we see that a homogeneous Borel set in $X$ is completely determined and topologically characterized by its Wadge class and its being first category or Baire.

4.4 COROLLARY. *There are exactly $\omega_1$ homogeneous zero-dimensional absolute Borel sets.*

PROOF. Theorem 4.3 and the results of van Engelen [**1**].

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