HOMOGENEOUS BOREL SETS

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Abstract. Topological characterizations of all zero-dimensional homogeneous absolute Borel sets are obtained; it turns out that there are $\omega_1$ such spaces. We use results from game theory—particularly, about Wadge classes.

1. Introduction and preliminaries. All spaces under discussion are separable and metrizable. We will assume that the reader is familiar with the main facts about absolute Borel sets (see [3 or 7]). Notation follows [7].

In this paper we describe and characterize all homogeneous Borel sets in the Cantor set that are not in $\Delta^0_3$ (i.e., they are not both $F_{\sigma\delta}$ and $G_{\delta\sigma}$). Together with [1], where all homogeneous Borel sets in $2^\omega$ of class $\Delta^0_3$ were determined, this yields a complete topological classification of all zero-dimensional homogeneous absolute Borel sets. Roughly, using the inductive definition of the non-self-dual Borel Wadge classes as given by Louveau [5], we show that the Wadge class of a non-$\Delta^0_3$ homogeneous Borel set in $2^\omega$ is non-self-dual and reasonably closed (for definitions, see below); then we can apply a theorem of Steel [8] to get what we want.

Let $Z$ be any space. If $\Gamma \subset \mathcal{P}(Z)$, then $\hat{\Gamma} = \{ A \subset Z : Z \setminus A \in \Gamma \}$, and $\Delta(\Gamma) = \Gamma \cap \hat{\Gamma}$. $\Gamma$ is called self-dual if $\hat{\Gamma} = \Gamma$. Mostly, we work inside the Cantor set $2^\omega$, denoted by $X$. Let $Q_i = \{ x \in X : \exists m \forall n \geq m : x_n = i \}$, for $i \in \{0,1\}$. Then $Q_0 \cong Q_1 \cong Q$, the space of rationals. If $x \notin Q_0 \cup Q_1$, then $x$ consists of blocks of zeros separated by blocks of ones; define $\phi : X \setminus (Q_0 \cup Q_1) \to X$ by $\phi(x)_n = 0$ if the $n$th block of zeros in $x$ has even length, and $\phi(x)_n = 1$ otherwise. Note that $\phi$ is continuous.

1.1 Definition (Steel [8]). (a) $\Gamma \subset \mathcal{P}(X)$ is a reasonably closed pointclass if $\phi^{-1}[A] \cup Q_0 \in \Gamma$ for each $A \in \Gamma$, and $f^{-1}[A] \in \Gamma$ for each $A \in \Gamma$ and each continuous $f : X \to X$. (b) $A \subset X$ is everywhere properly $\Gamma$ if for each open $U \neq \emptyset$ in $X$ we have $U \cap A \in \Gamma \setminus \hat{\Gamma}$.

1.2 Theorem (Steel [8]). If $\Gamma$ is a reasonably closed pointclass of Borel sets, and $A,B \subset X$ are everywhere properly $\Gamma$ and either both meager or both comeager, then $h[A] = B$ for some autohomeomorphism $h$ of $X$.

Now let $Z \in \{ X, \omega^\omega \}$. If $A,B \subset Z$, define $A \leq_w B$ if $A = f^{-1}[B]$ for some continuous $f : Z \to Z$. The Wadge class of $A$ is $[A] = \{ B \subset Z : B \leq_w A \}$, and $\Gamma \subset \mathcal{P}(Z)$ is a Borel Wadge class in $Z$ if $\Gamma = [A]$ for some Borel set $A$ in $Z$. Define [1].

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the Wadge ordering $\leq$ on the Wadge classes by $\Gamma_1 \leq \Gamma_2$ if $\Gamma_1 \subseteq \Gamma_2$, and $\Gamma_1 < \Gamma_2$ if $\Gamma_1 \leq \Gamma_2$ and $\Gamma_1 \neq \Gamma_2$. Using game theory, it can be shown that if $\Gamma_1$, $\Gamma_2$ are Borel Wadge classes in $\mathbb{Z}$, then $\Gamma_1 < \Gamma_2$, $\Gamma_1 \in \{ \Gamma_2, \tilde{\Gamma}_2 \}$, or $\tilde{\Gamma}_2 \leq \Gamma_1$. Furthermore, if $\Gamma_1 < \Gamma_2$, then also $\tilde{\Gamma}_1 < \tilde{\Gamma}_2$ (and hence $\tilde{\Gamma}_1 < \tilde{\Gamma}_2, \Gamma_2$). Thus, if we consider $\leq$ to be an ordering on pairs $\{ \Gamma, \tilde{\Gamma} \}$ of Borel Wadge classes, then $\leq$ becomes a linear ordering, and, in fact, a well-ordering in type $< \omega_2$ (see Wadge [10]; for some proofs, see [7]).

By van Wesep [9] the pattern of dual and non-self-dual Borel Wadge classes in the Wadge ordering on $\omega^\omega$ is as follows: The first element is $\{ \emptyset \}, \{ \omega^\omega \}$; a successor is self-dual if and only if its predecessor is not; at limit stages of cofinality $\omega$ stands a self-dual class; and at limit stages of cofinality $\omega_1$, a non-self-dual pair. Since we want to apply Theorem 1.2, we have to consider Borel Wadge classes in $\mathcal{X}$ instead of $\omega^\omega$. In [5] Louveau has given construction principles “from below” for the Borel Wadge classes in $\omega^\omega$; but analyzing his results and proofs, it can be seen that the same inductive definition can be given for the Borel Wadge classes in $\mathcal{X}$, with one exception: In the Borel Wadge ordering on $\mathcal{X}$, the limit stages of cofinality $\omega$ are occupied by non-self-dual pairs (see Theorem 1.5(b)).

The following definitions and theorem are all due to Louveau for $\omega^\omega$ instead of $\mathcal{X}$.

1.3 Definition. Let $\Gamma, \Gamma' \in \mathcal{P}(\mathcal{X})$, and let $A \subseteq \mathcal{X}$.

(a) $A \in D_\eta(\Sigma^0_\xi, \Gamma)$ if there is an increasing sequence $\langle A_\xi : \xi < \eta \rangle$ of $\Sigma^0_\xi$-sets such that $A = \bigcup \xi (A_\xi \setminus \bigcup_{\beta < \xi} A_\beta)$, where $\xi$ ranges over all even (odd) ordinals $\eta \text{ if } \eta \text{ is odd (even)}$.

(b) $A \in \text{Sep}(D_\eta(\Sigma^0_\xi, \Gamma), \Gamma')$ if $A = (A_1 \cap C) \cup (A_2 \setminus C)$ for some $C \subseteq D_\eta(\Sigma^0_\xi)$, $A_1 \subseteq \tilde{\Gamma}_1$, $A_2 \subseteq \tilde{\Gamma}_2$.

(c) $A \in \text{Biseq}(D_\eta(\Sigma^0_\xi, \Gamma), \Gamma', C)$ if $A = (A_1 \cap C_1) \cup (A_2 \cap C_2) \cup B \setminus (C_1 \cup C_2)$ for some disjoint $C_1, C_2 \subseteq D_\eta(\Sigma^0_\xi)$, and some $A_1 \subseteq \tilde{\Gamma}_1$, $A_2 \subseteq \tilde{\Gamma}_2$, $B \subseteq \Gamma'$.

(d) $A \in \text{SU}(\Sigma^0_\xi, \Gamma)$ if $A = \bigcup_n \omega(A_n \cap C_n)$ for some sequences $\langle C_n \rangle_n$ of pairwise disjoint $\Sigma^0_\xi$-sets, $\langle A_n \rangle_n$ of elements of $\Gamma$. The set $\bigcup_n \omega C_n$ is called the envelope of $A$.

(e) $A \in \text{SD}_\eta(\langle \Sigma^0_\xi, \text{SU}(\Sigma^0_\xi, \Gamma) \rangle, \Gamma')$ if $A = \bigcup_{\xi < \eta} (A_\xi \setminus \bigcup_{\beta < \xi} C_\beta) \cup B \setminus \bigcup_{\xi < \eta} C_\xi$ for some increasing sequences $\langle A_\xi : \xi < \eta \rangle$ of elements of $\text{SU}(\Sigma^0_\xi, \Gamma)$, and $\langle C_\xi : \xi < \eta \rangle$ of $\Sigma^0_\xi$-sets such that $A_\xi \subseteq C_\xi \subseteq A_{\xi+1}$ and $C_\xi$ is the envelope of $A_\xi$, and some $B \subseteq \Gamma'$.

In (c), (e), we omit $\Gamma'$ if $\Gamma' = \{ \emptyset \}$.

To simplify exposition, in writing, e.g., “$A \in \text{Sep}(D_\eta(\Sigma^0_\xi, \Gamma), \Gamma')$ if $A = (A_1 \cap C) \cup (A_2 \setminus C)$”, we always assume that the sets $A_1, A_2, C$ are chosen as in the above definition. Louveau now selects a certain subset $D$ of $\omega^\omega$, its elements being called descriptions, and for each $u \in D$, a non-self-dual Borel Wadge class $\Gamma_u$ is defined. Also, the type $t(u) \in \{ 0, 1, 2, 3 \}$ of a description $u$ is defined, and with each $u \in D$ of type 1, an element $\bar{u} \in D$ is associated, everything according to the following definition (where sometimes $v \in \omega^\omega$ is considered as a pair $\langle v_0, v_1 \rangle$ or a sequence $\langle v_n : n \in \omega \rangle$ of elements of $\omega^\omega$; $\emptyset \in \omega^\omega$ has all coordinates 0):

1.4 Definition. (a) $\emptyset \in D$, $\Gamma_0 = \{ \emptyset \}$, $t(\emptyset) = 0$.

(b) If $u = \xi^\chi \eta^\eta^\varphi \emptyset$, where $\xi \geq 1$, $\eta \geq 1$, then $u \in D$, $\Gamma_u = D_\eta(\Sigma^0_\xi)$. If $\eta$ is limit, then $t(u) = 2$; if $\eta = \eta_0 + 1$, then $t(u) = 1$, and $\bar{u} = \emptyset$ if $\eta_0 = \emptyset$, $\bar{u} = \xi^\chi \eta^\eta^\varphi \emptyset$, otherwise.
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(c) If \( u = \xi^2 \eta^u \), where \( \xi \geq 1, \eta \geq 1, u^* \in D, u^*(0) > \xi \), then \( u \in D, \Gamma_u = \text{Sep}(D(\eta^u, \Sigma^0_\xi), \Gamma_u), \) and \( t(u) = 3. \)

(d) If \( u = \xi^3 \eta^u \langle u_0, u_1 \rangle, \) where \( \xi \geq 1, \eta \geq 1, u_0, u_1 \in D, u_0(0) > \xi, u_1(0) \geq \xi \) or \( u_1 = 0, \) and \( \Gamma_{u_0} < \Gamma_{u_1}, \) then \( u \in D, \Gamma_u = \text{Bise}(D(\eta^u, \Sigma^0_\xi), \Gamma_u, \Gamma_u). \) If \( u_1 = 0 \) and \( \eta = \eta_0 + 1, \) then \( t(u) = 1, \) and \( \bar{u} = u_0 \) if \( \eta_0 = 0, \bar{u} = \xi^2 \eta^u u_0, \) otherwise. If \( u_1 = 0 \) and \( \eta \) is limit, then \( t(u) = 2. \) If \( u_1(0) > \xi, \) then \( t(u) = 3. \) If \( u_1(0) = \xi, \) then \( t(u) = t(u_1), \) and \( \bar{u} = \xi^3 \eta^u \langle u_0, u_1 \rangle \) if \( t(u_1) = 1. \)

(e) If \( u = \xi^4 \eta^u \langle u_0, u_1 \rangle \), where \( \xi \geq 1, \) each \( u_n \in D, \Gamma_{u_n} < \Gamma_{u_{n+1}}, \langle u_n(0) \rangle_n \) is nondecreasing and \( \sup u_n(0) > \xi, \) then \( u \in D, \Gamma_u = \text{SU}(\Sigma^0_\xi, \cup_{n \in \omega} \Gamma_{u_n}), t(u) = 2. \)

(f) If \( u = \xi^5 \eta^u \langle u_0, u_1 \rangle, \) where \( \xi \geq 1, \eta \geq 2, u_0, u_1 \in D, u_0(0) = \xi, u_0(1) = 4, u_1(1) \geq \xi \) or \( u_1 = 0, \) and \( \Gamma_{u_0} < \Gamma_{u_1}, \) then \( u \in D, \Gamma_u = \text{SD}(\Sigma^0_{u_0}, \Gamma_{u_0}, \Gamma_{u_1}). \) If \( u_1 = 0 \) then \( t(u) = 2. \) If \( u_1(0) > \xi, \) then \( t(u) = 3. \) If \( u_1(0) = \xi, \) then \( t(u) = u_1, \) and \( \bar{u} = \xi^5 \eta^u \langle u_0, u_1 \rangle \) if \( t(u_1) = 1. \)

1.5 Theorem. (a) If \( t(u) = 1 \) and \( u(0) = 1, \) then \( \Gamma_u < \Gamma_u, \) \( \text{and} \) \( \Delta(\Gamma_u) \) is the unique Borel Wadge class \( \Gamma \) such that \( \Gamma_u < \Gamma < \Gamma_u. \)

(b) If \( t(u) = 2 \) and \( u(0) = 1, \) then \( \Delta(\Gamma_u) = \bigcup_{n \in \omega} \Gamma_{u_n} \) for some strictly increasing sequence of described classes \( \langle \Gamma_{u_n} \rangle \) (for \( \omega^\omega \), Louveau has that \( \Delta(\Gamma_u) \) is the unique Borel Wadge class \( \Gamma \) such that \( \Gamma_{u_n} < \Gamma < \Gamma_u, \) for all \( n \in \omega \)).

(c) If \( u(0) > 1 \) or \( t(u) = 3, \) then \( \Delta(\Gamma_u) = \bigcup \{ \Gamma_{u_\alpha}: \alpha \in \omega_1 \} \) for some strictly increasing sequence of described classes \( \langle \Gamma_{u_\alpha} \rangle. \)

From this theorem, it is easily deduced that \( \left\{ \Gamma_u: u \in D \right\} \cup \left\{ \Gamma_u^*: u \in D \right\} \cup \left\{ \Delta(\Gamma_u): u \in D, t(u) = 1, u(0) = 1 \right\} \) is the set of all Borel Wadge classes in \( X. \)

2. Closure properties. The statements in the following lemma were proved in Louveau [5] for Borel Wadge classes in \( \omega^\omega \). However, the proofs work for \( X \) as well.

2.1 Lemma. Let \( u \in D, u(0) = \xi. \)

(a) \( \text{SU}(\Sigma^0_\xi, \Gamma_u) = \Gamma_u, \) and if \( \eta < \xi, \) then \( \text{SU}(\Sigma^0_\eta, \Gamma_u) = \Gamma_u. \)

(b) \( \Gamma_u \) and \( \Gamma_u^* \) are closed under union and intersection with a \( \Delta^0_\xi \)-set.

(c) If \( u(1) = 4, \) then \( \Gamma_u \) is closed under union with a \( \Sigma^0_\xi \)-set.

(d) If \( t(u) = 3, \) then \( \Gamma_u \) and \( \Gamma_u^* \) are closed under union with a \( \Sigma^0_\xi \)-set and under intersection with a \( \Pi^0_\xi \)-set.

(e) If \( u = \xi^3 \eta^u \langle u_0, u_1 \rangle, \) and \( A \in \Gamma_u, \) then there exist \( C \in \Sigma^0_\xi \) and \( B \in \Gamma_{u_0}, \) such that \( A = (A \cap C) \cup (B \setminus C), \) and both \( A \cap C \) and \( C \setminus A \) are in \( \text{Bise}(\Sigma^0_\xi, \Gamma_{u_0}). \)

(f) If \( t(u) = 1, \) then \( \Gamma_u = \text{Bise}(\Sigma^0_\xi, \Gamma_u), \) with \( \bar{u} \) as defined in 1.4.

(g) If \( t(u) = 2, \) then \( \Gamma_u = \text{SU}(\Sigma^0_\xi, \cup_{n \in \omega} \Gamma_{u_n}) \) for some strictly increasing sequence \( \langle \Gamma_{u_n} \rangle_n \) of described classes with \( u_n(0) > \xi \) for all \( n \in \omega. \)

In this section we will prove more closure properties of some classes \( \Gamma_u, \) \( \Gamma_u^* \) similar to (a)–(d) in the preceding lemma.

2.2 Lemma. If \( \Delta^0_\xi \subseteq \Gamma_u \) and \( u(0) \geq 2, \) then \( \Gamma_u \) is closed under intersection with a \( \Pi^0_\xi \)-set and under union with a \( \Sigma^0_\xi \)-set; hence so is \( \Gamma_u^*. \)
PROOF. If not, there is a minimal class $\Gamma_u$ for which it fails. By 2.1(d) the lemma holds if $t(u) = 3$, and by 2.1(b) if $u(0) \geq 3$, so we have $u(0) = 2$ and $t(u) \in \{1, 2\}$.

Case 1. $t(u) = 1$. By 2.1(f), $\Gamma_u = \text{Bise}p(\Sigma^0_2, \Gamma_{u0})$. Since $\Delta^0_2 \subset \Gamma$, also $\Delta^0_2 \subset \Gamma_u$ (otherwise $\Gamma_u = \Delta^0_3$, but $\Gamma_u$ is non-self-dual). In Definition 1.4 we see that in (b) we have $\Gamma_u \supset \Delta^0_3$, so we must be in (d) or (f), whence $u(0) \geq 2$. Since $\Gamma_u \cap \Gamma_u \subset \Gamma_u$, we have $\Gamma_u \leq \Gamma_u$, so by minimality of $\Gamma_u$, $\Gamma_u$ has the described closure properties. Let $A \subset \Gamma_u$, say $A = (A_1 \cap C_1) \cup (A_2 \cap C_2)$. If $F \subset \Sigma^0_2$, then $A_1 \cap F \subset \Gamma_u$, $A_2 \cap F \subset \Gamma_u$, so $A \cap F = (A_1 \cap F \cap C_1) \cup (A_2 \cap F \cap C_2) \subset \Gamma_u$. If $G \subset \Sigma^0_2$, let $C_1^*, C_2^*$ reduce $C_1 \cup G$, $C_2 \cup G$. Then $A \cup G = ((A_1 \cup G) \cap C_1^*) \cup ((A_2 \cup G) \cap C_2^*) \subset \Gamma_u$.

Case 2. $t(u) = 2$. By 2.1(g), $\Gamma_u = \text{SU}(\Sigma^0_2, \bigcup_{n \in w} \Gamma_{u_n})$. If each $\Gamma_{u_n} \subset \Delta^0_3$, then $\Gamma_u \subset \text{SU}(\Sigma^0_2, \Delta^0_3)$. Now if $v \in 3^1 \cdot 1^1 \cdot 0^1$, then $\Gamma_v = \Sigma^0_2$, so by 2.1(a), we have $\text{SU}(\Sigma^0_2, \Delta^0_3) \subset \text{SU}(\Sigma^0_2, \Sigma^0_3) \cup \text{SU}(\Sigma^0_2, \Pi^0_3) = \Sigma^0_3 \cap \Pi^0_3 = \Delta^0_3$, so $\Gamma_u \subset \Delta^0_3$, a contradiction. Thus we conclude that $\Delta^0_3 \subset \Gamma_u$, for some $n$, and hence $\Delta^0_3 \subset \Gamma_{u_m}$ for all $m \geq n$. Since $u_m(0) \geq 2$ and $\Gamma_{u_n} < \Gamma_u$, each $\Gamma_{u_m}$ has the described closure properties; now proceed as in Case 1. □

2.3 LEMMA. If $u(0) \geq 3$, or $u(0) = 2$ and $t(u) = 3$, then $\Gamma_u$ is closed under union with a $\Pi^0_2$-set.

PROOF. If the lemma fails, there is a minimal $\Gamma_u$ for which it does. Since the lemma is true if $u(0) \geq 3$ by 2.1(b), we have $u(0) = 2$ and $t(u) = 3$. Let $F \subset \Pi^0_2$.

Case 1. $u(1) = 2$, so $\Gamma_u = \text{Sep}(D_{\eta}(\Sigma^0_2), \Gamma_{u0})$, $u^*(0) > 2$. If $A \subset \Gamma_u$, say $A = (A_1 \cap C) \cup (A_2 \setminus C)$, then $A_1 \cap F \subset \Gamma_u$, $A_2 \cap F \subset \Gamma_u$, by 2.1(b), so $A \cap F = ((A_1 \cap F) \cap C) \cup (A_2 \cap F) \cap C \subset \Gamma_u$.

Case 2. $u(1) = 3$, so $\Gamma_u = \text{Bise}p(D_{\eta}(\Sigma^0_2), \Gamma_{u0}, \Gamma_{u1})$, $u_{0}(0) > 2$, and $u_{1}(0) > 2$ or $(u_{1}(0) = 2$ and $t(u_{1}) = 3$). If $A \subset \Gamma_u$, say

$$A = (A_1 \cap C_1) \cup (A_2 \cap C_2) \cup B \setminus (C_1 \cup C_2),$$

then $A_1 \cap F \subset \Gamma_u$, $A_2 \cap F \subset \Gamma_u$, by 2.1(b), and $B \cap F \subset \Gamma_u$, by 2.1(b) if $u_{1}(0) > 2$, and by minimality of $\Gamma_u$ if $u_{1}(0) = 2$. So

$$A \cap F = ((A_1 \cap F) \cap C_1) \cup ((A_2 \cap F) \cap C_2) \cup (B \cap F) \setminus (C_1 \cup C_2) \subset \Gamma_u.$$

Case 3. $u(1) = 5$, so $\Gamma_u = \text{SD}_{\eta}(\Sigma^0_2, \Gamma_{u0}, \Gamma_{u1})$, $u_{0}(0) = 2$, $u_{1}(0) = 4$, and $u_{1}(0) > 2$ or $(u_{1}(0) = 2$ and $t(u_{1}) = 3)$. Let $A \subset \Gamma_u$, say

$$A = \bigcup_{\xi < \eta} \left( A_\xi \cup \bigcup_{\beta < \xi} C_{\beta} \right) \cup B \setminus \bigcup_{\xi < \eta} C_{\xi}.$$ 

Since $X \setminus F \subset \Sigma^0_2$, and $u_{0}(0) = 2$, by 2.1(a) we have $A_\xi \cap (X \setminus F) \subset \Gamma_{u0}$, and it is easily verified that the envelop of $A_\xi \cap (X \setminus F)$ is $C_{\xi} \cap (X \setminus F)$. Also $B \cap F \subset \Gamma_{u1}$ as in Case 2. So

$$A \cap F = \bigcup_{\xi < \eta} \left( (A_\xi \cap (X \setminus F)) \setminus \bigcup_{\beta < \xi} (C_{\beta} \cap (X \setminus F)) \right) \cup (B \cap F) \setminus \bigcup_{\xi < \eta} (C_{\xi} \cap (X \setminus F)) \subset \Gamma_u.$$

□
2.4 Corollary. If \( u(0) \geq 3 \), or \( u(0) = 2 \) and \( t(u) = 3 \), then \( SU(\Sigma^0_3, \tilde{T}_u) = \tilde{T}_u \).

Proof. If \( A \in SU(\Sigma^0_3, \tilde{T}_u) \), say \( A = \bigcup_{n \in \omega} (A_n \cap C_n) \), then \( X \setminus A = \bigcup_{n \in \omega} (C_n \cap X \setminus A_n) \cup X \setminus \bigcup_{n \in \omega} C_n \). Now \( X \setminus A \in \Gamma_u \), so \( \bigcup_{n \in \omega} (C_n \cap X \setminus A_n) \in SU(\Sigma^0_3, \Gamma_u) = \Gamma_u \) by 2.1(a), and \( X \setminus \bigcup_{n \in \omega} C_n \in \Pi^0_2 \), thus, \( X \setminus A \in \Gamma_u \) by 2.3, whence \( A \in \tilde{T}_u \). □

3. Existence of homogeneous Borel sets. Some notation: If we apply the operations of 1.4 in a space \( Z \), then we obtain classes \( \Gamma_u(Z) \) (so \( \Gamma_u = \Gamma_u(X) \)). Inductively, it is easily shown that if \( Z \subset X \), then \( A \in \Gamma_u(Z) \) if and only if \( A = B \cap Z \) for some \( B \in \Gamma_u \), and similarly for \( \tilde{T}_u \) (for the cases of 1.4(d), (e), use the reduction property).

We write \( Z_1 \simeq Z_2 \) if \( Z_1 \) is homeomorphic to \( Z_2 \), \( h: Z_1 \approx Z_2 \) if \( h \) is a homeomorphism.

3.1 Lemma. If \( \Delta^0_3 \subset \Gamma_u \) and \( u(0) \geq 2 \), and if \( B \subset X, A \in \Gamma_u, B \approx A, \) then \( B \in \Gamma_u \), and similarly for \( \tilde{T}_u \).

Proof. Let \( f: A \approx B \). By Lavrentieff's theorem (see [4 or 2]), there exist \( \Pi^0_2 \)-sets \( G, H \) in \( X \) with \( A \subset G, B \subset H \), and a homeomorphism \( \tilde{f}: G \to H \) extending \( f \).

Since \( A \in \Gamma_u \), also \( A \in \Gamma_u(G) \), so \( B \in \Gamma_u(H) \), say \( B = \tilde{B} \cap H \) with \( \tilde{B} \in \Gamma_u \). Since \( H \in \Pi^0_2 \), by 2.2 we have \( B \in \Gamma_u \). The proof for \( \tilde{T}_u \) is analogous. □

3.2 Definition. If \( \Delta^0_3 \subset \Gamma_u \) and \( u(0) \geq 2 \), then a zero-dimensional space \( Y \) is everywhere properly \( \mathcal{P}_u \) (resp. \( \mathcal{H}_u \)) if some copy of \( Y \) in \( X \) is everywhere properly \( \Gamma_u \) (resp. \( \tilde{T}_u \)).

Note that from 3.1 it follows that if \( Y \) is everywhere properly \( \mathcal{P}_u \) (resp. \( \mathcal{H}_u \)), then each dense embedding of \( Y \) in \( X \) is everywhere properly \( \Gamma_u \) (resp. \( \tilde{T}_u \)), and that “everywhere properly \( \mathcal{P}_u \)” and “everywhere properly \( \mathcal{H}_u \)” are topological properties.

3.3 Lemma. If \( \Delta^0_3 \subset \Gamma_u \) and \( u(0) \geq 2 \), let \( \mathcal{V}_u^0 \) (\( \mathcal{V}_u^1 \)), \( \mathcal{L}_u^0 \) (\( \mathcal{L}_u^1 \)) be the classes of all zero-dimensional spaces that are, respectively, everywhere properly \( \mathcal{P}_u \) and first category (Baire), everywhere properly \( \mathcal{H}_u \) and first category (Baire). Then up to homeomorphism, each class contains at most one space, and if it exists, this space is homogeneous.

Proof. To prove the first part of the lemma, it suffices to show that \( \Gamma_u \) and \( \tilde{T}_u \) are reasonably closed. For then if, e.g., \( A, B \in \mathcal{V}_u^0 \), and \( \bar{A}, \bar{B} \) are copies of \( A, B \) in \( X \) that are everywhere properly \( \Gamma_u \), then \( \bar{A} \) and \( \bar{B} \) are meager, so we can apply Theorem 1.2; the other cases are similar. So let \( \phi \) be as in §1, and put \( P = X \setminus (Q_0 \cup Q_1) \). If \( A \in \Gamma_u \), then clearly \( \phi^{-1}[A] \in \Gamma_u(P) \). Hence for some \( A' \in \Gamma_u \), we have \( \phi^{-1}[A] = A' \cap P \). Since \( P \in \Pi^0_2 \), \( \phi^{-1}[A] \in \Gamma_u \) by 2.2, and hence \( \phi^{-1}[A] \cup Q_0 \in \Gamma_u \) by 2.2, since \( Q_0 \in \Sigma^0_2 \). The proof for \( \tilde{T}_u \) is the same.

For the second part of the lemma, note that if \( A \) is in one of the defined classes, then any nonempty open-and-closed subset of \( A \) is in the same class since \( u(0) \geq 2 \) (use 2.1(a)), and hence it is homeomorphic to \( A \); such a space is called strongly homogeneous, and it is not hard to show that any strongly homogeneous zero-dimensional space is homogeneous (see e.g. [6]). □
Thus, if $Y$ is in one of the above classes, then $Y$ is a homogeneous space that is topologically characterized by the properties describing the class.

We now determine which of the classes are nonempty.

3.4 Lemma. If $\Delta_3^0 \subseteq \Gamma_u$ and $u(0) \geq 2$, then $\mathscr{V}_u^0$ and $\mathscr{Z}_u^1$ are nonempty.

Proof. Let $Z \in \tilde{\Gamma}_u \setminus \Gamma_u$, and let $O = \bigcup \{U \in Z, U \in \Gamma_u\}$. Then for some $U_n \in \Gamma_u$ with $U_n$ open in $Z$, we have $O = \bigcup_{n \in \omega} U_n$. Let $\tilde{U}_n$ be open in $X$ with $\tilde{U}_n \cap Z = U_n$, and let $(V_n)_n$ reduce $(\tilde{U}_n)_n$. Then $V_n \cap Z = V_n \cap \tilde{U}_n \cap Z = V_n \cap U_n$, $U_n \in \Gamma_u$, $u(0) \geq 2$, $V_n \in \Sigma^1_0$, so by 2.1(b), $V_n \cap Z \in \Gamma_u$. So $O = \bigcup_{n \in \omega} \{V_n \cap Z\} \in SU(\Sigma^1_0, \Gamma_u) = \Gamma_u$ by 2.1(a). Put $\tilde{Z} = Z \setminus O$; then $\tilde{Z} \neq \emptyset$ since $Z \notin \Gamma_u$. Since $\tilde{Z} = Z \setminus \bigcup_{n \in \omega} V_n$, and $X \setminus \bigcup_{n \in \omega} V_n = \Pi^0_1 \subseteq \Delta^0_2$, we have $\tilde{Z} \in \tilde{\Gamma}_u$ by 2.1(b). We claim that no nonempty open subset $U$ of $\tilde{Z}$ is in $\Gamma_u$. Indeed, if $U \in \Gamma_u$, choose $\tilde{U}$ open in $X$ with $\tilde{U} \cap Z = U$; then

$$\tilde{U} \cap Z = \left( \left( X \setminus \bigcup_{n \in \omega} V_n \right) \cap (\tilde{U} \cap \tilde{Z}) \right) \cup \bigcup_{n \in \omega} ((\tilde{U} \cap V_n) \cap (V_n \cap Z))$$

by 2.1(a), so $\tilde{U} \cap Z \subset O$, contradicting $\emptyset \neq U \subset (\tilde{U} \cap Z) \setminus O$. Now let $Z'$ be a densely embedded copy of $\tilde{Z}$ in $X$ (which exists since $\tilde{Z}$ contains no isolated points), and put $Y = X \setminus Z'$. Also, let $Q$ be a countable dense subset of $X$, and put $Y_u^0 = Q \times Y \subset X \times X$. We identify $X \times X$ with $X$, and claim that $Y_u^0 \in \mathscr{V}_u^0$. First note that, by 3.1, $Z' \in \tilde{\Gamma}_u$, hence $Y \in \Gamma_u$ and $\{q\} \times Y \in \Gamma_u$, so

$$Q \times Y = \bigcup_{q \in Q} (\{q\} \times Y \cap \{q\} \times X) \in SU(\Sigma^1_0, \Gamma_u) = \Gamma_u.$$

Now if $V$ is a nonempty open subset of $X \times X$, then $V \cap Y_u^0 \neq \emptyset$, say $(q, x) \in V \cap Y_u^0$. Then $U = (\{q\} \times Y) \cap V$ is a nonempty open subset of $\{q\} \times Y$, and also it is closed in $V \cap Y_u^0$. So if $V \cap Y_u^0$ were in $\tilde{\Gamma}_u$, then also $U \in \tilde{\Gamma}_u$ by 2.1(b); but then $Y$ contains a nonempty open subset $U'$ with $U' \in \tilde{\Gamma}_u$, say $U' = U \cap Y$ with $U$ open in $X$. Then $U \cap (X \setminus U) = \tilde{U} \cap Z'$ is a nonempty open subset of $Z'$ which is in $\Gamma_u$ since $\tilde{U} \in \Sigma^1_0$, $X \setminus U \in \Gamma_u$, $u(0) \geq 2$, a clear contradiction. Thus, $Y_u^0$ is everywhere properly $\Gamma_u$, and obviously it is first category; so $Y_u^0 \in \mathscr{V}_u^0$. Arguing as above, it is easily seen that $(X \times X) \setminus Y_u^0 \in \mathscr{Z}_u^1$. $\square$

If we try to prove that $\mathscr{Z}_u^0$ and $\mathscr{V}_u^1$ are nonempty by replacing $\Gamma_u$ in the above proof by $\tilde{\Gamma}_u$, then we see that we need $SU(\Sigma^0_2, \tilde{\Gamma}_u) = \tilde{\Gamma}_u$; as we shall see in Lemma 3.6, this is not always the case. However, from 2.4 we see that the following holds:

3.5 Lemma. If $\Delta_3^0 \subseteq \Gamma_u$, and $u(0) \geq 3$ or $(u(0) = 2$ and $t(u) = 3)$, then $\mathscr{Z}_u^0$ and $\mathscr{V}_u^1$ are nonempty. $\square$

In fact, if $Z_u^1$ and $Q$ are as in the proof of 3.4, then $Z_u^0 = Q \times Z_u^1 \in \mathscr{Z}_u^0$, and $X^3 \setminus Z_u^0 \in \mathscr{V}_u^1$.

3.6 Lemma. If $\Delta_3^0 \subseteq \Gamma_u$, $u(0) = 2$, and $t(u) \in \{1, 2\}$, then $\mathscr{V}_u^1 = \emptyset = \mathscr{Z}_u^0$. 

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Proof. If \( Z \in \mathcal{B}^0_u \) is densely embedded in \( X \), then \( X \setminus Z \in \mathcal{B}^1_u \) (argue as in the proof of 3.4), so it suffices to show that \( \mathcal{B}^1_u = \emptyset \). We will prove that if \( A \subset X \) is everywhere properly \( \Gamma_u \), then \( A \) is first category. First take \( t(u) = 2 \). By 2.1(g), \( \Gamma_u = \text{SU}(\eta_u, \omega) \) so we can write \( A = \bigcup_{n \in \omega} (A_n \cap C_n) \). Let \( C_n = \bigcup_{m \in \omega} C_n^m \), with \( C_n^m \in \Pi^0_1 \); then if \( A_n \in \Gamma_{u_n} \), also \( C_n^m \cap A = C_n^m \cap A_n \in \Gamma_{u_n} \) since \( u_k(0) \geq 2 \), \( C_n^m \in \Delta^0_2 \), using 2.1(b). If \( U \neq \emptyset \) is open in \( A \), say \( U = \tilde{U} \cap A \) with \( \tilde{U} \) open in \( X \), and if \( U \subset C_n^m \cap A_n \), then \( U = \tilde{U} \cap C_n^m \cap A_n \in \Gamma_{u_n} \subset \tilde{U} \) since \( \Gamma_{u_n} < \Gamma_u \), a contradiction. So \( C_n^m \cap A \) is closed and nowhere dense in \( A \), whence

\[
A = \bigcup_{n,m \in \omega} (C_n^m \cap A)
\]

is first category. If \( t(u) = 1 \), note that since \( \Delta^0_3 \subset \Gamma_u \), we have \( \bar{u} \neq 0 \) whence \( \bar{u}(0) \geq 2 \). Since \( \Gamma_u = \text{Bisep}(\Sigma^0_2, \Gamma_u) \) by 2.1(f), we can argue as above. \( \square \)

4. The Wadge class of a homogeneous Borel set. In this section we show that if \( Y \) is a homogeneous zero-dimensional absolute Borel set, and \( Y \notin \Delta^0_3 \), then \( Y \subset \mathcal{B}^0_u \cup \mathcal{B}^1_u \cup \mathcal{B}^0_u \cup \mathcal{B}^1_u \) for some \( u \in D, u(0) \geq 2, \Delta^0_3 \subset \Gamma_u \).

4.1 Lemma. Let \( Y \) be a homogeneous Borel set in \( X \) with \( Y \notin \Delta^0_3 \). Let \( \Gamma_u \) be the least described class such that \( A \in \Gamma_u \) for some nonempty open subset \( A \) of \( Y \). Then \( \Delta^0_3 \subset \Gamma_u \) and \( u(0) \geq 2 \).

Proof. If \( \Delta^0_3 \subset \Gamma_u \), then \( \Gamma_u \cap \Gamma' \subset \Delta^0_3 \) whence \( A \in \Delta^0_3 \). Let \( x \in A \), and by homogeneity of \( Y \), let \( h_x : Y \rightarrow Y \) be such that \( h_x(x) = y \). Take a countable subcovering \( \{ h_x[A] : n \in \omega \} \) of the open covering \( \{ h_x[A] : y \in Y \} \) of \( Y \), and let \( U_n \) be open in \( X \) such that \( U_n \cap Y = h_x[A] \). If \( \langle V_n \rangle_n \) reduces \( \langle U_n \rangle_n \), then

\[
V_n \cap Y = V_n \cap U_n \cap Y = V_n \cap h_x[A] \subset \Delta^0_3,
\]

so \( Y = \bigcup_{n \in \omega} (V_n \cap Y) \subset \text{SU}(\Sigma^0_1, \Delta^0_3) = \Delta^0_3 \) (see the proof of 2.2 Case 2), a contradiction. So \( \Delta^0_3 \subset \Gamma_u \).

Now assume \( u(0) = 1 \).

Case 1. \( t(u) = 2 \). By 2.1(g), \( \Gamma_u = \text{SU}(\eta_u, \omega) \). If \( A \subset \Gamma_u \), say \( A = \bigcup_{n \in \omega} (A_n \cap C_n) \), then some \( C_n \subset A_n \subset C_n \subset A \) is a nonempty open subset of \( A \), hence of \( Y \); but \( A_n \subset \Gamma_{u_n} \), \( u_k(0) > 1 \), so \( C_n \subset A_n \subset \Gamma_{u_n} \) by 2.1(a), contradicting minimality of \( \Gamma_u \) since \( \Gamma_{u_k} < \Gamma_u \). If \( A \subset \Gamma_u \), then \( X \setminus A = \bigcup_{n \in \omega} (A_n \cap C_n) \), so \( A = \bigcup_{n \in \omega} (C_n \setminus X \setminus A_n) \cup X \setminus \bigcup_{n \in \omega} C_n \). Since \( \Gamma_{u_k} < \Gamma_{u_k+1} \), each \( X \setminus A_n \subset \bigcup_{k \in \omega} \Gamma_{u_k} \), so if some \( C_n \subset X \setminus A_n \neq \emptyset \), we obtain a contradiction as above; thus, \( A = X \setminus \bigcup_{n \in \omega} C_n \subset \Pi^0_1 \subset \Delta^0_3 \), which is impossible.

Case 2. \( u = 1^\wedge 1 \wedge \eta + 1^\wedge 0 \). Then \( \Gamma_{u'} = \text{D}_{u+1}(\eta_u) \), so \( A \in \Delta^0_3 \).

Case 3. \( u = 1^\wedge 2 \wedge \eta * u \). Then \( \Gamma_u = \text{Sep}(\text{D}_{u}(\eta_u), \Gamma_{u'}), u^*(0) \geq 2 \). If \( A \subset \Gamma_u \), then \( A = (A_1 \cap C) \cup (A_2 \setminus C), C \in \text{D}_{u}(\eta_u) \). Let \( C = U_{\zeta} (C_{\zeta} \setminus \bigcup_{\beta < \zeta} C_{\beta}) \) as in Definition 1.3(a).

(i) If \( C_{\zeta} \cap A = \emptyset \) for all \( \zeta < \eta \), then \( A = A_2 \setminus U_{\zeta < \eta} C_{\zeta} \). Since \( A_2 \subset \Gamma_{u'} \) and \( X \setminus \bigcup_{\zeta < \eta} C_{\zeta} \subset \Pi^0_1 \subset \Delta^0_2 \), we have \( A \subset \Gamma_{u^*} \) by 2.1(b); but \( \Gamma_{u^*} < \Gamma_u \), contradicting minimality of \( \Gamma_u \).
(ii) Let \( \alpha < \eta \) be minimal with \( C_\alpha \cap A \neq \emptyset \). If \( \alpha \) and \( \eta \) are both even or both odd, then \( C_\alpha \setminus \bigcup_{\eta < \alpha} C_\beta \subset X \setminus C \), so \( C_\alpha \cap A = C_\alpha \cap A_2 \setminus C \). Since \( C_\alpha \cap X \setminus C \in \Delta_0^2 \), \( C_\alpha \cap A \in \Gamma_u^* \), as above, and \( C_\alpha \cap A \) is a nonempty open subset of \( Y \). If \( \alpha \) is even and \( \eta \) is odd, or conversely, then \( C_\alpha \setminus \bigcup_{\eta < \alpha} C_\beta \subset C \), so \( C_\alpha \cap A = C_\alpha \cap C \cap A_1 \in \Gamma_u^* \), which again is impossible. If \( A \in \Gamma_u^* \), then \( X \setminus A = (A_1 \cap C) \cup (A_2 \setminus C) \), so \( A = ((X \setminus A_1) \cap C) \cup (X \setminus A_2) \setminus C \). Put \( A_1 = X \setminus A \), \( A_2 = X \setminus A_2 \), and argue as above.

Case 4. \( u = 1^3 \cdot 1^* \wedge 1^* \langle u_0, 0 \rangle \), \( u_0(0) \geq 2 \). Then \( \Gamma_u = \text{Biseq}(\Sigma_0^0, \Gamma_u) \) by 1.4(d) and 2.1(f). If \( A \in \Gamma_u \), then \( A = \left( A_1 \cap C_1 \right) \cup \left( A_2 \cap C_2 \right) \). Now \( C_1 \cap A = C_1 \cap A_1 \), and either \( A_1 \cap C_1 \) or \( A_2 \cap C_2 \) is nonempty. Since \( C_1 \in \Sigma_1^0 \subset \Delta_0^2 \), we obtain from 2.1(b) that \( A_1 \cap C_1 \in \Gamma_u^* \), \( A_2 \cap C_2 \in \Gamma_u^* \), so we have a contradiction.

If \( A \in \Gamma_u^* \), then \( X \setminus A = (A_1 \cap C_1) \cup (A_2 \cap C_2) \), so \( A = (C_1 \cap X \setminus A_1) \cup (C_2 \cap X \setminus A_2) \cup (C_1 \cap C_2) \). Since \( X \setminus A_1 \in \Gamma_u^* \) and \( X \setminus A_2 \in \Gamma_u^* \), we must have \( C_1 \cap X \setminus A_1 = \emptyset \) by the above argument, so \( A = X \setminus (C_1 \cap C_2) \in \Pi_0^0 \subset \Delta_0^2 \), another contradiction.

Case 5. \( u = 1^3 \cdot 3 \wedge 1^* \langle u_0, 0, 0 \rangle \), \( u_0(0) \geq 2 \), \( u_0(1) \geq 1 \). Then \( \Gamma_u = \text{Biseq}(\Sigma_1^0, \Gamma_u) \), where \( \Gamma_u = \text{Sep}(D_\eta(\Sigma_0^0, \Gamma_u^0, \Gamma_u)) \). It suffices to show that \( \Gamma_u \) and \( \Gamma_u^* \) are closed under intersection with a \( \Sigma_1^0 \)-set, for then we can copy the proof of Case 4, replacing \( \Gamma_u^0 \) by \( \Gamma_u \). For \( \Gamma_u^* \), this follows from 2.1(a); for \( \Gamma_u \), it is equivalent to \( \Gamma_u \) being closed under union with a \( \Pi_0^0 \)-set, and this is proved exactly as 2.3 Case 1.

Case 6. \( u = 1^3 \cdot 3 \wedge 1^* \langle u_0, 1, 1 \rangle \), \( u_0(0) \geq 2 \), \( u_0(1) \geq 1 \). Then \( \Gamma_u = \text{Biseq}(D_\eta(\Sigma_1^0, \Gamma_u^0, \Gamma_u)) \). If \( A \in \Gamma_u \), then by 2.1(e) we can write \( A = (A \cap C) \cup B \setminus C \) for some \( C \in \Sigma_1^0 \), \( B \in \Gamma_u^1 \) such that \( A \cap C \in \text{Biseq}(D_\eta(\Sigma_1^0, \Gamma_u)) \). Since \( A \cap C \) is open in \( A \), and since it is easily checked that \( \text{Biseq}(D_\eta(\Sigma_1^0, \Gamma_u)) < \Gamma_u \) (use that \( X \in \Gamma_u \) since \( u_0(1) \geq 1 \)), we must have \( A \cap C = \emptyset \), whence \( A = B \setminus C \). But \( B \in \Gamma_u^0 \), \( X \setminus C \in \Delta_0^2 \), \( u_0(0) \geq 2 \), so \( B \setminus C \in \Gamma_u^0 \) by 2.1(b), so \( A \in \Gamma_u^0 < \Gamma_u^* \), a contradiction. If \( A \in \Gamma_u^* \), then again by 2.1(e), we have \( X \setminus A = ((X \setminus A) \cap C) \cup B \setminus C \) for some \( C \in \Sigma_1^0 \), \( B \in \Gamma_u^1 \) with \( C \setminus (X \setminus A) = C \cap A \in \text{Biseq}(D_\eta(\Sigma_0^0, \Gamma_u)) \). Thus \( A = (A \cap C) \cup (X \setminus B) \setminus C \), whence, as above, \( A = (X \setminus B) \setminus C \). But \( X \setminus B \in \Gamma_u \setminus \Gamma_u^0 \), so we again obtain a contradiction.

Case 7. \( u = 1^5 \cdot 3 \wedge 1^* \langle u_0, 0, 1 \rangle \), \( u_0(0) = 1 \), \( u_0(1) = 4 \), \( u_0(1) \geq 1 \). Then \( \Gamma_u = \text{SD}_\eta(\Sigma_1^0, \Gamma_u) \), \( \Gamma_u^0 \). If \( A \in \Gamma_u \), then \( A = \bigcup_{\eta < \xi} (A_\xi \setminus \bigcup_{\eta < \xi} C_\beta) \cup B \setminus \bigcup_{\eta < \xi} C_\beta \). Put \( C = \bigcup_{\eta < \xi} C_\beta \). Again, it is easily checked that \( \text{SD}_\eta(\Sigma_1^0, \Gamma_u^0) < \Gamma_u \) since \( C \in \Sigma_1^0 \), and \( C \cap A = \bigcup_{\eta < \xi} (A_\xi \setminus \bigcup_{\eta < \xi} C_\beta) \in \text{SD}_\eta(\Sigma_1^0, \Gamma_u^0) \), we have \( C \cap A = \emptyset \), so \( A = B \setminus C \). By 2.1(c), \( \Gamma_u^0 \) is closed under union with a \( \Sigma_1^0 \)-set, so \( \Gamma_u^0 \) is closed under intersection with a \( \Pi_0^0 \)-set. Since \( B \in \Gamma_u^0 \subset \Gamma_u \), we have \( A \in \Gamma_u^0 < \Gamma_u^* \). If \( A \in \Gamma_u^* \), and \( X \setminus A = \bigcup_{\eta < \xi} (A_\xi \setminus \bigcup_{\eta < \xi} C_\beta) \cup B \setminus \bigcup_{\eta < \xi} C_\beta \), then use Lemma 2.1 to show that

\[
A = \bigcup_{\xi < \eta} \left( \left( \bigcup_{\eta < \xi} C_\beta \right) \setminus \bigcup_{\eta < \xi} C_\beta \right) \cup (X \setminus B) \setminus \bigcup_{\xi < \eta} C_\xi
\]

\[
\in \text{SD}_\eta(\Sigma_1^0, \Gamma_u, \Gamma_u^0)
\]

and argue as above.
4.2 Lemma. Let \( Y \) be a homogeneous zero-dimensional absolute Borel set with \( Y \notin \Delta^0_3 \). Then for some \( u \in D \) with \( \Delta^0_3 \subset \Gamma_u \) and \( u(0) \geq 2 \), we have \( Y \in \mathcal{B}_u^0 \cup \mathcal{B}_u^1 \cup \mathcal{X}_u^0 \cup \mathcal{X}_u^1 \).

Proof. Embed \( Y \) densely in \( X \), and let \( \Gamma \) be the least Borel Wadge class such that \( A \in \Gamma \cup \hat{\Gamma} \) for some nonempty open subset \( A \) of \( Y \). If \( \Gamma \) is self-dual, then \( \Gamma = \Delta(\Gamma_u) \) for some \( v \in D \) with \( t(v) = 1 \), \( v(0) = 1 \) (see §1). But then \( \Gamma_v \) is the least described class such that \( B \in \Gamma_v \cup \hat{\Gamma}_v \) for some nonempty open \( B \) in \( Y \), contradicting 4.1. So \( \Gamma \) is non-self-dual, say \( \Gamma = \Gamma_u \), and by 4.1, \( \Delta^0_3 \subset \Gamma_u \) and \( u(0) \geq 2 \). Let \( x \in A \), and for each \( y \in Y \), let \( h_y : Y \rightarrow Y \) be such that \( h_y(x) = y \). Let \( \{ h_{y_n}[A] : n \in \omega \} \) be a countable subcovering of the open covering \( \{ h_y[A] : y \in Y \} \) of \( Y \), and let \( U_n \) be open in \( X \) such that \( U_n \cap Y = h_{y_n}[A] \). If \( h_{y_n}[A] \) reduces \( U_n \), then

\[ V_n \cap Y = V_n \cap U_n \cap Y = V_n \cap h_{y_n}[A]. \]

Now \( h_{y_n}[A] \in \Gamma_u \) (resp. \( \hat{\Gamma}_u \)) if \( A \in \Gamma_u \) (resp. \( \hat{\Gamma}_u \)) by 3.1. Since \( V_n \in \Sigma^0_1 \) and \( u(0) \geq 2 \), we have \( V_n \cap Y \in \Gamma_u \) (resp. \( \hat{\Gamma}_u \)) by 2.1(b). So \( Y \in \text{SU}(\Sigma^0_1, \Gamma_u) = \Gamma_u \) (resp. \( Y \in \text{SU}(\Sigma^0_1, \hat{\Gamma}_u) = \hat{\Gamma}_u \)) by 2.1(a). A similar argument shows that if \( B \neq \emptyset \) is open in \( X \), and \( B \cap Y \) were in \( \Gamma_u \) (resp. \( \hat{\Gamma}_u \)), then we would have \( Y \in \Gamma_u \) (resp. \( Y \in \hat{\Gamma}_u \)), since \( B \cap Y \neq \emptyset \), so \( Y \in \Delta(\Gamma_u) \). Hence by 1.5(c), \( Y \in \Gamma_v \) for some \( \Gamma_v < \Gamma_u \), contradicting minimality of \( \Gamma_v \). Thus \( Y \) is everywhere properly \( \Gamma_u \) (resp. everywhere properly \( \hat{\Gamma}_u \)), and since a homogeneous space is either first category or Baire, the result follows. 

Remark. In the above lemma, we in fact have that \( Y \in \mathcal{B}_u^0 \cup \mathcal{B}_u^1 \) if \( [Y] = \Gamma_u \), and \( Y \in \mathcal{X}_u^0 \cup \mathcal{X}_u^1 \) if \( [Y] = \hat{\Gamma}_u \).

We can now formulate our main theorem. Let \( D_0 = \{ u \in D : \Delta^0_3 \subset \Gamma , u(0) \geq 2 \} \), and \( D_1 = \{ u \in D_0 : u(0) \geq 3 \text{ or } t(u) = 3 \} \). By 3.4 and 3.5, for each \( u \in D_0 \), there are elements \( Y_u^0, Z_u^1 \) in \( \mathcal{B}_u^0, \mathcal{X}_u^0 \), respectively, and for each \( u \in D_1 \), there are elements \( Y_u^1, Z_u^0 \) in \( \mathcal{B}_u^1, \mathcal{X}_u^1 \), respectively.

4.3 Theorem. Up to homeomorphism, \( \{ Y_u^0, Z_u^1 : u \in D_0 \} \cup \{ Y_u^1, Z_u^0 : u \in D_1 \} \) consists precisely of all homogeneous zero-dimensional absolute Borel sets outside \( \Delta^0_3 \).

Proof. Apply 3.3, 3.6, and 4.2. 

Thus, from the remark following 4.2 and from the above theorem, we see that a homogeneous Borel set in \( X \) is completely determined and topologically characterized by its Wadge class and its being first category or Baire.

4.4 Corollary. There are exactly \( \omega_1 \) homogeneous zero-dimensional absolute Borel sets.

Proof. Theorem 4.3 and the results of van Engelen [1].

References


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