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MAPS WHICH PRESERVE ANR'S

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ABSTRACT. It is shown that maps preserve ANR's and LC^n-property if they satisfy a certain movability condition in the fiber shape theory. This generalizes the known results of hereditary shape equivalences to a non cell-like case.

1. Introduction. Spaces are assumed to be metrizable and ANR's are ones for metric spaces. Suppose $f\colon X \to Y$ is a proper onto map ($f^{-1}(B)$ is compact for each compact $B \subset Y$) and X is an ANR. It is a long-standing problem to determine conditions on f under which Y is an ANR. G. Kozlowski **[K]** proved that if f is a hereditary shape equivalence then Y is an ANR. In **[Y_1]** we introduced the notion of movability for maps and proved **[Y_1**, Theorem 1.2] that f is a hereditary shape equivalence iff f is a CE-map and movable. Here an onto map $f\colon X \to Y$ is said to be *movable* provided for some (eq. any) ANR M containing X as a closed subset the following holds:

(*) *For each neighborhood U of $f^{-1} = \bigcup\{\{y\} \times f^{-1}(y)\colon y \in Y\}$ in $Y \times M$ there exists a neighborhood V of f^{-1} in U such that for each neighborhood W of f^{-1} in V there exists a homotopy $h\colon V \times [0,1] \to U$ such that $h_0 = \mathrm{id}$, $h_1(V) \subset W$, $ph_t = p$ $(0 \leqslant t \leqslant 1)$, where $p\colon Y \times M \to Y$ is the projection.*

In addition, if we can take the homotpy h so that $h_t|_{f^{-1}} = \mathrm{id}$ $(0 \leqslant t \leqslant 1)$, then we say f is *strongly* movable.

The purpose of this note is to show that this movability assumption on f is sufficient to ensure that Y is an ANR.

THEOREM. *Suppose $f\colon X \to Y$ is a movable map.*
(1) *If X is an ANR then so is Y.*
(2) *If X is locally n-connected (LC^n) then so is Y.*

In **[K]**, it was also proved that a CE-map with an ANR domain is a hereditary shape equivalence iff the range is an ANR. Our theorem, combined with **[CD_1**, Proposition 3.6], **[Y_1**, Corollary 4.4], yields the following version.

COROLLARY. *Suppose $f\colon X \to Y$ is a proper onto map and X is a separable, locally compact ANR. Then f is (strongly) movable iff Y is an ANR and f is completely movable.*

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As for the definition of the complete movability, refer to [**CD**$_1$], and for the other related topics, refer to [**Y**$_1$].

REMARK. (i) In the above corollary, the complete movability of f implies the approximate homotopy lifting property for all n-cells ($n \geqslant 0$) [**CD**$_1$, Theorem 3.3, Proposition 3.6] and also that each fiber of f is an FANR. Therefore Y is LC^∞ by [**CD**$_2$, Theorem 3.4]. However, in general, Y is *not* necessarily an ANR, because there exists a CE (hence completely movable) map from the Hilbert cube to a compactum which is not a shape equivalence [**T**].

(ii) In [**Y**$_2$], it is shown that any movable map does *not* raise dimension. Therefore, if $f\colon X \to Y$ is a completely movable map and X is an n-dimensional, locally compact ANR, then Y is an ANR iff $\dim Y \leqslant n$ [**Y**$_2$, Corollary 3.5].

2. Proof of Theorem. Suppose $f\colon X \to Y$ is a movable map and M and N are ANR's which contain X and Y as closed subsets respectively. Let $p\colon N \times M \to N$ denote the projection and let ρ be a metric on N. For each neighborhood U of f^{-1} in $N \times M$, we define $U|_Y = U \cap Y \times M$. First we have a lemma.

LEMMA 1. *Let W_i ($i \geqslant 0$) be open neighborhoods of f^{-1} in $N \times M$. Then there exist an open neighborhood U of f^{-1} in $Y \times M$, open neighborhoods V_i ($i \geqslant 0$) of f^{-1} in $N \times M$ and a map $h\colon V_0 \times [0,\infty) \to W_0$ such that $V_{i+1} \subset V_i$, $U = V_i|_Y$, $h_0 = \mathrm{id}$, $ph_t = p$, $h(V_i \times [i,\infty)) \subset W_i$ ($i \geqslant 0$).*

PROOF. We may assume $W_{i+1} \subset W_i$ ($i \geqslant 0$). Since f is movable, there exist open neighborhoods U_i ($i \geqslant 0$) of f^{-1} in $Y \times M$ and homotopies $g^i\colon U_i \times [0,1] \to U_{i-1}$ ($i \geqslant 1$) such that $U_{i+1} \subset U_i \subset W_i|_Y$ ($i \geqslant 0$), $g_0^i = \mathrm{id}$, $g_1^i(U_i) \subset U_{i+1}$, $pg_t^i = p$ ($i \geqslant 1$, $0 \leqslant t \leqslant 1$). Let $U = U_1$ and define $g\colon U \times [0,\infty) \to W_0$ by

$$g(y,x,t) = g^n\left(g_1^{n-1} \circ \cdots \circ g_1^1(y,x), t-(n-1)\right) \in U_{n-1}$$
$$\text{for } (y,x) \in U,\ n-1 \leqslant t \leqslant n,\ n \geqslant 1.$$

Take an open neighborhood V_0 of f^{-1} in W_0 with $V_0|_Y = U$. Since M is an ANR, using the Borsuk homotopy extension theorem or its proof [**H**, p. 117], inductively we can find maps $h^n\colon V_0 \times [0,n] \to W_0$ ($n \geqslant 1$) and open neighborhoods V_n ($n \geqslant 1$) of U in V_0 such that $h^{n+1}|_{V_0 \times [0,n]} = h^n$, $h_0^n = \mathrm{id}$, $h^n|_{U \times [0,n]} = g|_{U \times [0,n]}$, $ph_s^n = p$ ($0 \leqslant s \leqslant n$), $h^n(V_i \times [i,n]) \subset W_i$ ($0 \leqslant i \leqslant n$). The desired map h is obtained by piecing h^n ($n \geqslant 1$) together.

PROOF OF THEOREM (1). Since the ANR $f^{-1} \approx X$ is closed in $N \times M$, there exists a retraction $r\colon W_0 \to f^{-1}$ from some open neighborhood of f^{-1} in $N \times M$. Since $pr|_{f^{-1}} = p|_{f^{-1}}$, we can find open neighborhoods W_i ($i \geqslant 1$) of f^{-1} in W_0 such that $\rho(pr(z), p(z)) < 1/i$ ($z \in W_i$, $i \geqslant 1$). Applying Lemma 1 to W_i ($i \geqslant 0$), we obtain U, V_i ($i \geqslant 0$) and h as in Lemma 1.

Let $y_0 \in Y$. Take a point $x_0 \in f^{-1}(y_0)$ and an open neighborhood K of y_0 in N such that $K \times \{x_0\} \subset V_0$. We will show that there exists a map $k\colon K \to Y$ such that $k|_{K \cap Y} = \mathrm{id}_{K \cap Y}$. Then $K \cap Y$ is an ANR neighborhood of y_0 in Y since $k|_{k^{-1}(K \cap Y)}$ is a retraction from an ANR $k^{-1}(K \cap Y)$ onto $K \cap Y$. This implies that Y is a local ANR, hence an ANR [**H**, p. 68].

Define a map $s\colon K \to V_0$ by $s(y) = (y, x_0)$ $(y \in K)$. Since $s(K \cap Y) \subset U \subset V_i$ $(i \geqslant 0)$, there exist closed neighborhoods K_i $(i \geqslant 0)$ of $K \cap Y$ in K such that $K_0 = K$, $K_{i+1} \subset \operatorname{Int} K_i$, $\bigcap K_i = K \cap Y$ and $s(K_i) \subset V_i$ $(i \geqslant 1)$. Take a function $\lambda\colon K - Y \to [0, \infty)$ with $\lambda(K_i - Y) \subset [i, \infty)$ and define $k\colon K \to Y$ by

$$k(y) = \begin{cases} y, & y \in K \cap Y, \\ prh(s(y), \lambda(y)), & y \in K - Y. \end{cases}$$

If $y \in K_i - Y$, $i \geqslant 1$, then $h(s(y), \lambda(y)) \in W_i$ and by the choice of W_i, $\rho(k(y), y) < 1/i$. The continuity of k follows from this observation. This completes the proof.

We proceed to the proof of Theorem (2) and assume $f^{-1} \approx X$ is LC^n. If $\mathscr{U}$ is an open cover of f^{-1} in $N \times M$, then two maps g, $g'\colon P \to f^{-1}$ are said to be $\mathscr{U}$-near if for each $x \in P$ there exists $U \in \mathscr{U}$ such that $g(x)$, $g'(x) \in U$. The next lemma follows from [**H**, p. 156, Theorem 4.1] and will play the same role as the retraction $r\colon W_0 \to f^{-1}$ in the preceding proof.

LEMMA 2. *Let $\mathscr{U}_i$ $(i \geqslant 0)$ be a sequence of open coverings of f^{-1} in $N \times M$. Then there exist open neighborhoods W_i $(i \geqslant 0)$ of f^{-1} in $N \times M$ such that if $P = \bigcup\{P_i\colon i \geqslant 0\}$ is an $(n + 1)$-dimensional locally compact polyhedron, P_i is a compact subpolyhedron of P, $P_i \subset \operatorname{Int} P_{i+1}$ $(i \geqslant 0)$ and $g\colon P \to W_0$ is a map with $g(P_i - \operatorname{Int} P_{i-1}) \subset W_i$ $(i \geqslant 0)$, then there exists a map $g'\colon P \to f^{-1}$ such that g and g' are $\mathscr{U}_i$-near on $P_i - \operatorname{Int} P_{i-1}$ for $i \geqslant 0$.*

PROOF OF THEOREM (2). To see Y is LC^n, let $y_0 \in Y$ and let L_0 be any neighborhood of y_0 in Y. Take open neighborhoods K_0, K_1 of y_0 in N such that $K_0 \cap Y = L_0$, $\operatorname{Cl} K_1 \subset K_0$ ($\operatorname{Cl} K_1$ is the closure of K_1 in N).

For each $i \geqslant 1$ take an open covering $\mathscr{U}_i$ of $N \times M$ which refines $\mathscr{U}_0 = \{(N - \operatorname{Cl} K_1) \times M, K_0 \times M\}$ and such that for each $U_i \in \mathscr{U}_i$, $\operatorname{diam} p(U_i) < 1/i$. There exist open neighborhoods W_i $(i \geqslant 0)$ of f^{-1} in $N \times M$ as in Lemma 2. Then there exist U, V_i $(i \geqslant 0)$ and h as in Lemma 1. Take a point $x_0 \in f^{-1}(y_0)$ and open neighborhoods $K_3 \subset K_2$ of y_0 in K_1 such that $K_2 \times \{x_0\} \subset V_0$ and the inclusion $K_3 \subset K_2$ is nullhomotopic (note that the ANR N is locally contractible).

Let $L = K_3 \cap Y$. We will show that any map $\alpha\colon S^k \to L$ from the k-sphere S^k $(0 \leqslant k \leqslant n)$ has an extension $\beta\colon B^{k+1} \to L_0$ over the $(k + 1)$-ball B^{k+1}.

Since α is nullhomotopic in K_2, α extends to a map $\gamma\colon B^{k+1} \to K_2$. Define $s\colon B^{k+1} \to V_0$ by $s(z) = (\gamma(z), x_0)$, $z \in B^{k+1}$. Since $s(S^k) \subset U \subset V_i$ $(i \geqslant 0)$, there exist compact subpolyhedra P_i $(i \geqslant 0)$ of $\mathring{B}^{k+1}$ (the interior of B^{k+1}) such that $\mathring{B}^{k+1} = \bigcup P_i$, $P_i \subset \operatorname{Int} P_{i+1}$, $s(B^{k+1} - \operatorname{Int} P_i) \subset V_{i+1}$ $(i \geqslant 0)$. Take a function $\lambda\colon \mathring{B}^{k+1} \to [0, \infty)$ such that $\lambda(\mathring{B}^{k+1} - \operatorname{Int} P_i) \subset [i + 1, \infty)$ $(i \geqslant 0)$. Define $g\colon \mathring{B}^{k+1} \to W_0$ by $g(z) = h(s(z), \lambda(z))$. Then $g(P_i - \operatorname{Int} P_{i-1}) \subset W_i$ $(i \geqslant 0)$. By the choice of W_i $(i \geqslant 0)$, we have a map $g'\colon \mathring{B}^{k+1} \to f^{-1}$ such that g and g' are $\mathscr{U}_i$-near on $P_i - \operatorname{Int} P_{i-1}$ $(i \geqslant 0)$. Note that $pg = \gamma$, $pg'(\mathring{B}^{k+1}) \subset L_0$, $\rho(pg'(z), \gamma(z)) < 1/i$ $(z \in P_i - \operatorname{Int} P_{i-1}$, $i \geqslant 1)$. Finally define the map $\beta\colon B^{k+1} \to L_0$ by $\beta|_{S^k} = \alpha$, $\beta|_{\mathring{B}^{k+1}} = pg'$. The continuity of β follows from the above observation. This completes the proof.

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