MAPS WHICH PRESERVE ANR'S

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Abstract. It is shown that maps preserve ANR's and LC\"-property if they satisfy a certain movability condition in the fiber shape theory. This generalizes the known results of hereditary shape equivalences to a non cell-like case.

1. Introduction. Spaces are assumed to be metrizable and ANR's are ones for metric spaces. Suppose $f: X \to Y$ is a proper onto map ($f^{-1}(B)$ is compact for each compact $B \subset Y$) and $X$ is an ANR. It is a long-standing problem to determine conditions on $f$ under which $Y$ is an ANR. G. Kozlowski [K] proved that if $f$ is a hereditary shape equivalence then $Y$ is an ANR. In [Y1] we introduced the notion of movability for maps and proved [Y1, Theorem 1.2] that $f$ is a hereditary shape equivalence iff $f$ is a CE-map and movable. Here an onto map $f: X \to Y$ is said to be movable provided for some (eq. any) ANR $M$ containing $X$ as a closed subset the following holds:

(*) For each neighborhood $U$ of $f^{-1}$ in $Y \times M$ there exists a neighborhood $V$ of $f^{-1}$ in $U$ such that for each neighborhood $W$ of $f^{-1}$ in $V$ there exists a homotopy $h: V \times [0,1] \to U$ such that $h_0 = \text{id}$, $h_1(V) \subset W$, $ph_t = p$ ($0 \leq t \leq 1$), where $p: Y \times M \to Y$ is the projection.

In addition, if we can take the homotopy $h$ so that $h_t|_{f^{-1}} = \text{id}$ ($0 \leq t \leq 1$), then we say $f$ is strongly movable.

The purpose of this note is to show that this movability assumption on $f$ is sufficient to ensure that $Y$ is an ANR.

Theorem. Suppose $f: X \to Y$ is a movable map.
(1) If $X$ is an ANR then so is $Y$.
(2) If $X$ is locally n-connected (LC\") then so is $Y$.

In [K], it was also proved that a CE-map with an ANR domain is a hereditary shape equivalence iff the range is an ANR. Our theorem, combined with [CD1, Proposition 3.6], [Y1, Corollary 4.4], yields the following version.

Corollary. Suppose $f: X \to Y$ is a proper onto map and $X$ is a separable, locally compact ANR. Then $f$ is (strongly) movable iff $Y$ is an ANR and $f$ is completely movable.

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As for the definition of the complete movability, refer to [CD₁], and for the other related topics, refer to [Y₁].

Remark. (i) In the above corollary, the complete movability of \( f \) implies the approximate homotopy lifting property for all \( n \)-cells \((n \geq 0)\) [CD₁, Theorem 3.3, Proposition 3.6] and also that each fiber of \( f \) is an FANR. Therefore \( Y \) is LC\( ^\infty \) by [CD₂, Theorem 3.4]. However, in general, \( Y \) is not necessarily an ANR, because there exists a CE (hence completely movable) map from the Hilbert cube to a compactum which is not a shape equivalence [T].

(ii) In [Y₂], it is shown that any movable map does not raise dimension. Therefore, if \( f: X \to Y \) is a completely movable map and \( X \) is an \( n \)-dimensional, locally compact ANR, then \( Y \) is an ANR iff \( \dim Y \leq n \) [Y₂, Corollary 3.5].

2. Proof of Theorem. Suppose \( f: X \to Y \) is a movable map and \( M \) and \( N \) are ANR’s which contain \( X \) and \( Y \) as closed subsets respectively. Let \( p: N \times M \to N \) denote the projection and let \( \rho \) be a metric on \( N \). For each neighborhood \( U \) of \( f^{-1} \) in \( N \times M \), we define \( U|_Y = U \cap Y \times M \). First we have a lemma.

**Lemma 1.** Let \( W_i \) (\( i \geq 0 \)) be open neighborhoods of \( f^{-1} \) in \( N \times M \). Then there exist an open neighborhood \( U \) of \( f^{-1} \) in \( Y \times M \), open neighborhoods \( V_i \) (\( i \geq 0 \)) of \( f^{-1} \) in \( N \times M \) and a map \( h: V_0 \times [0, \infty) \to W_0 \) such that \( V_{i+1} \subset V_i \), \( U = V_1|_Y \), \( h_0 = \text{id} \), \( ph_i = \rho \), \( h(V_i \times [i, \infty)) \subset W_i \) (\( i \geq 0 \)).

**Proof.** We may assume \( W_{i+1} \subset W_i \) (\( i \geq 0 \)). Since \( f \) is movable, there exist open neighborhoods \( U_i \) (\( i \geq 0 \)) of \( f^{-1} \) in \( Y \times M \) and homotopies \( g^i: U_i \times [0, 1] \to U_{i-1} \) (\( i \geq 1 \)) such that \( U_{i+1} \subset U_i \subset W_i|_Y \) (\( i \geq 0 \)), \( g_0^i = \text{id} \), \( g^i(U_i) \subset U_{i+1} \), \( pg^i = \rho \) \((i \geq 1, 0 \leq t \leq 1)\). Let \( U = U_1 \) and define \( g: U \times [0, \infty) \to W_0 \) by

\[
g(y, x, t) = g^n(g_1^{-1} \circ \cdots \circ g_1(y, x), t - (n - 1)) \in U_{n-1}
\]

for \((y, x) \in U, n - 1 \leq t \leq n, n \geq 1\).

Take an open neighborhood \( V_0 \) of \( f^{-1} \) in \( W_0 \) with \( V_0|_Y = U \). Since \( M \) is an ANR, using the Borsuk homotopy extension theorem or its proof [H, p. 117], inductively we can find maps \( h^s: V_0 \times [0, n] \to W_0 \) (\( n \geq 1 \)) and open neighborhoods \( V_n \) (\( n \geq 1 \)) of \( U \) in \( V_0 \) such that \( h^{n+1}|_{V_0 \times [0, n]} = h^n \), \( h^n_0 = \text{id} \), \( h^n_{U \times [0, n]} = g^i_{U \times [0, n]} \), \( ph^n_i = \rho \) \((0 \leq s \leq n)\), \( h^n(V_i \times [i, n]) \subset W_i \) \((0 \leq i \leq n)\). The desired map \( h \) is obtained by piecing \( h^n \) \((n \geq 1)\) together.

**Proof of Theorem (1).** Since the ANR \( f^{-1} = X \) is closed in \( N \times M \), there exists a retraction \( r: W_0 \to f^{-1} \) from some open neighborhood of \( f^{-1} \) in \( N \times M \). Since \( pr|_{f^{-1}} = p|_{f^{-1}} \), we can find open neighborhoods \( W_i \) (\( i \geq 1 \)) of \( f^{-1} \) in \( W_0 \) such that \( \rho(pr(z), p(z)) < 1/i \) \((z \in W_i, i \geq 1)\). Applying Lemma 1 to \( W_i \) (\( i \geq 0 \)), we obtain \( U_i \subset V_i \) (\( i \geq 0 \)) and \( h \) as in Lemma 1.

Let \( y_0 \in Y \). Take a point \( x_0 \in f^{-1}(y_0) \) and an open neighborhood \( K \) of \( y_0 \) in \( N \) such that \( K \times \{x_0\} \subset V_0 \). We will show that there exists a map \( k: K \to Y \) such that \( k|_{K \cap Y} = \text{id}_{K \cap Y} \). Then \( K \cap Y \) is an ANR neighborhood of \( y_0 \) in \( Y \) since \( k^{-1}|_{K \cap Y} \) is a retraction from an ANR \( k^{-1}(K \cap Y) \) onto \( K \cap Y \). This implies that \( Y \) is a local ANR, hence an ANR [H, p. 68].
Define a map \( s: K \to V_0 \) by \( s(y) = (y, x_0) \) \((y \in K)\). Since \( s(K \cap Y) \subseteq U \subseteq V_i \) \((i \geq 0)\), there exist closed neighborhoods \( K_i \) \((i \geq 0)\) of \( K \cap Y \) in \( K \) such that \( K_0 = K, K_{i+1} \subseteq \text{Int} K_i, \cap K_i = K \cap Y \) and \( s(K_i) \subseteq V_i \) \((i \geq 1)\). Take a function \( \lambda: K - Y \to [0, \infty) \) with \( \lambda(K_i - Y) \subseteq [i, \infty) \) and define \( k: K \to Y \) by

\[
k(y) =
\begin{cases}
  y, & y \in K \cap Y, \\
  \text{prh}(s(y), \lambda(y)), & y \in K - Y.
\end{cases}
\]

If \( y \in K_i - Y, i \geq 1 \), then \( h(s(y), \lambda(y)) \in W_i \) and by the choice of \( W_i, \rho(k(y), y) < 1/i \). The continuity of \( k \) follows from this observation. This completes the proof.

We proceed to the proof of Theorem (2) and assume \( f^{-1} = X \) is \( LC^n \). If \( \mathcal{U} \) is an open cover of \( f^{-1} \) in \( N \times M \), then two maps \( g, g': P \to f^{-1} \) are said to be \( \mathcal{U} \)-near if for each \( x \in P \) there exists \( U \in \mathcal{U} \) such that \( g(x), g'(x) \in U \). The next lemma follows from [H, p. 156, Theorem 4.1] and will play the same role as the retraction \( r: W_0 \to f^{-1} \) in the preceding proof.

**Lemma 2.** Let \( \mathcal{U}_i \) \((i \geq 0)\) be a sequence of open coverings of \( f^{-1} \) in \( N \times M \). Then there exist open neighborhoods \( W_i \) \((i \geq 0)\) of \( f^{-1} \) in \( N \times M \) such that if \( P = \bigcup \{ P_i: i \geq 0 \} \) is an \((n + 1)\)-dimensional locally compact polyhedron, \( P_i \) is a compact subpolyhedron of \( P \), \( P_i \subseteq \text{Int} P_{i+1} \) \((i \geq 0)\) and \( g: P \to W_0 \) is a map with \( g(P_i - \text{Int} P_{i-1}) \subseteq W_i \) \((i \geq 0)\), then there exists a map \( g': P \to f^{-1} \) such that \( g \) and \( g' \) are \( \mathcal{U}_i \)-near on \( P_i - \text{Int} P_{i-1} \) for \( i \geq 0 \).

**Proof of Theorem (2).** To see \( Y \) is \( LC^n \), let \( y_0 \in Y \) and let \( L_0 \) be any neighborhood of \( y_0 \) in \( Y \). Take open neighborhoods \( K_0, K_1 \) of \( y_0 \) in \( N \) such that \( K_0 \cap Y = L_0 \), \( \text{Cl} K_1 \subseteq K_0 \) (\( \text{Cl} K_1 \) is the closure of \( K_1 \) in \( N \)).

For each \( i \geq 1 \) take an open covering \( \mathcal{U}_i \) of \( N \times M \) which refines \( \mathcal{U}_0 = \{(N - \text{Cl} K_1) \times M, K_0 \times M\} \) and such that for each \( U \in \mathcal{U}_i \), \( \diam \rho(U_i) < 1/i \). There exist open neighborhoods \( W_i \) \((i \geq 0)\) of \( f^{-1} \) in \( N \times M \) as in Lemma 2. Then there exist \( U, V_i \) \((i \geq 0)\) and \( h \) as in Lemma 1. Take a point \( x_0 \in f^{-1}(y_0) \) and open neighborhoods \( K_3, K_2 \) of \( y_0 \) in \( K_1 \) such that \( K_2 \times \{x_0\} \subseteq V_0 \) and the inclusion \( K_3 \subseteq K_2 \) is nullhomotopic (note that the ANR \( N \) is locally contractible).

Let \( L = K_3 \cap Y \). We will show that any map \( \alpha: S^k \to L \) from the k-sphere \( S^k \) \((0 \leq k \leq n)\) has an extension \( \beta: B^{k+1} \to L_0 \) over the \((k + 1)\)-ball \( B^{k+1} \).

Since \( \alpha \) is nullhomotopic in \( K_2 \), \( \alpha \) extends to a map \( \gamma: B^{k+1} \to K_2 \). Define \( s: B^{k+1} \to V_0 \) by \( s(z) = (\gamma(z), x_0) \), \( z \in B^{k+1} \). Since \( s(S^k) \subseteq U \subseteq V_i \) \((i \geq 0)\), there exist compact subpolyhedra \( P_i \) \((i \geq 0)\) of \( B^{k+1} \) (the interior of \( B^{k+1} \)) such that \( B^{k+1} = \bigcup P_i \), \( P_i \subseteq \text{Int} P_{i+1} \), \( s(B^{k+1} - \text{Int} P_i) \subseteq V_{i+1} \) \((i \geq 0)\). Take a function \( \lambda: B^{k+1} \to [0, \infty) \) such that \( \lambda(B^{k+1} - \text{Int} P_i) \subseteq [i + 1, \infty) \) \((i \geq 0)\). Define \( g: B^{k+1} \to W_0 \) by \( g(z) = h(s(z), \lambda(z)) \). Then \( g(P_i - \text{Int} P_{i-1}) \subseteq W_i \) \((i \geq 0)\). By the choice of \( W_i \) \((i \geq 0)\), we have a map \( g': B^{k+1} \to f^{-1} \) such that \( g \) and \( g' \) are \( \mathcal{U}_i \)-near on \( P_i - \text{Int} P_{i-1} \) \((i \geq 0)\). Note that \( pg = \gamma, pg(\hat{B}^{k+1}) \subseteq L_0 \), \( \rho(pg(z), \gamma(z)) < 1/i \) \((z \in P_i - \text{Int} P_{i-1}, i \geq 1)\). Finally define the map \( \beta: B^{k+1} \to L_0 \) by \( \beta|_{B^{k+1}} = \alpha \), \( \beta|_{B^{k+1} - \text{Int} P_{i-1}} = pg' \). The continuity of \( \beta \) follows from the above observation. This completes the proof.
REFERENCES


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