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AN INTERSECTION HOMOLOGY OBSTRUCTION TO IMMERSIONS

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ABSTRACT. Given an immersion of a pseudomanifold in a manifold, certain Thom operations are naturally defined in intersection homology. This is used to obtain nonimmersion results for singular spaces.

0. The intersection homology groups of M. Goresky and R. MacPherson **[GM1, GM2, G]** are among the few topological invariants of singular spaces which are not homotopy invariants. Their invention opens up exciting new possibilities for the study of singular spaces.

It is natural to ask whether the intersection homology invariants can detect obstructions to geometric problems, such as questions of immersions or embeddings. Our aim is to show that for immersions, they can: Our main result, Theorem 2.1, states that if the singular space X immerses in a manifold, then certain Thom operations (depending on the dimensions involved) are definable for intersection homology. On the other hand, these operations are not always naturally definable. For example, the operation $\theta_1^{\mathbf{Z}/2}$ on $H_3(\Sigma\mathbf{R}P^2)$ is nontrivial but cannot lift to the zero perversity (cohomology), since $I_0H_2(\Sigma\mathbf{R}P^2) = H^1(\Sigma\mathbf{R}P^2) = 0$. (Geometrically this corresponds to the fact that the class $\theta_1^{\mathbf{Z}/2}[\Sigma\mathbf{R}P^2] = \Sigma\theta_1^{\mathbf{Z}/2}[\mathbf{R}P^2] = \Sigma\mathbf{R}P^1$ cannot be pushed off the singular set.)

We remark here that the above destroys the hope of calculating intersection homology as the ordinary homology of a suitable space, in general, since the Thom operations are natural. In particular, since small resolutions may be used to calculate intersection homology, the above gives obstructions to the existence of such resolutions.

My search for such obstructions began with the following question, posed by A. Haefliger: Can one find a class of singularities and embedding results for these, perhaps involving the intersection homology groups? C. Weber suggested one should first look at immersions, since the intersection homology sheaves involve knowing only a neighborhood of the diagonal.

In considering the relation of intersection homology to the problem of embedding, it is natural to look at the double point cycle of a map in general position. The

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double points will meet the stratum with perversity 0. However, the double points are not closed, in general. Their closure contains the singularities of the map, and it is possible that these singularities be contained entirely in the singular stratum. Thus in general, the double point cycle may not be allowed for any perverity. But for an immersion, it is allowed in perversity 0.

It is clear therefore that there is a geometric obstruction to immersion given by the double point cycle not lying in perversity 0. The close relationship between the double point cycle and the Thom operations (cf. [**M**, **H**]) leads to the present algebraic formulation.

Finally, the above shows that if $f\colon X^n \to M^m$ is an immersion, taking double points of cycles defines a map (not a homomorphism) $I_pH_i(X) \to I_pH_{2i-m}(X)$ (mod 2 coefficients, say) which must vanish if f is regularly homotopic to an embedding. This gives new obstructions to homotoping an immersion to an embedding.

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1. The operation of Thom. The discussion below is an adapted version of [**T**] (and may also be found in [**H**]).

Let R be a ring with unit, r an integer and set $R_\varepsilon = R/[1 + (-1)^r)R$. Recall that the Steenrod squaring operations are maps $\mathrm{Sq}^r\colon H^i(\ ; R) \to H^{i+r}(\ ; R_\varepsilon)$.

BPL will denote the classifying space for stable pl bundles. Let (X, A) be a pair of finite CW complexes and let $\varphi \in [X, \mathrm{BPL}]$ be a homotopy class of maps. (If $2 \neq 0$ in R we assume $\varphi_*\colon \pi_1(X) \to \pi_1(\mathrm{BPL}) = \mathbf{Z}/2\mathbf{Z}$ is the zero homomorphism.)

Choose a pl manifold V^N and a homotopy embedding of (X, A) in V, i.e. a homotopy equivalence $g\colon (X, A) \tilde{\to} (X', A')$, where X', A' are compact pl subspaces contained in the interior of V, such that $\varphi = [\psi \circ f]$; here $\psi\colon V \to \mathrm{BPL}$ classifies the stable tangent bundle of V and $f\colon X \tilde{\to} V$ is the composite $X \tilde{\to} X' \to V$.

Let D be the inverse of the Poincaré-Lefschetz duality map

$$\cap[V]\colon H^*(V\setminus A', V\setminus X') \to H_{N-*}(X', A').$$

Then the Thom operations $\theta_r^R(X, A; \varphi)$ are defined by $g_*^{-1}D^{-1}\mathrm{Sq}^rDg_*$, i.e. so that the diagram

$$\begin{array}{ccc}
H_n(X, A; R) & \xrightarrow{\theta_r^R(X, A;\varphi)} & H_{n-r}(X, A; R_\varepsilon) \\
\downarrow \wr\, g_* & & \downarrow \wr\, g_* \\
H_n(X', A'; R) & & H_{n-r}(X', A'; R_\varepsilon) \\
\uparrow \wr\, D^{-1} = \cap[V] & & \uparrow \wr\, D^{-1} = \cap[V] \\
H^{N-n}(V\setminus A', V\setminus X'; R) & \xrightarrow{\mathrm{Sq}^r} & H^{N-n+r}(V\setminus A', V\setminus X'; R_\varepsilon)
\end{array}$$

commutes.

Proposition 1.1 (Thom). *$\theta_r^R(X, A; \varphi)$ depends only on the homotopy type of (X, A) and φ, i.e. it is independent of the choice of the dimension N, the manifold V, and homotopy embedding. θ is natural and commutes with ∂_*, i.e. if $f\colon (X, A) \to (Y, B)$ is a continuous map, then $f_*\theta_r^R(X, A; f^*\varphi) = \theta_r^R(Y, B; \varphi) \circ f_*$ and $\partial_*\theta_r^R(X, A; \varphi) = \theta_r^R(A; \varphi|A)\partial_*$.*

PROOF. See [**T** or **H**].

NOTATION. In case φ is the class of the constant map, we write θ_r^R or θ_r in place of $\theta_r^R(X, A; \varphi)$. In this case we may take V to be Euclidean space.

COROLLARY 1.2. *θ_r is a stable homology operation, S-dual to* Sq^r.

PROOF. The naturality statement for $\theta_r^R(X, A; \varphi)$ implies that of θ_r, since f^* (constant) = constant. Now a natural homology operation commuting with ∂_* is stable.

To see that θ_r is S-dual to Sq^r, let $X' \subset S^N$ and $X^* \subset S^N \setminus X'$, a deformation retract, i.e. an N-dual. Put $A = \varnothing$. The composition

$$H_i(X) \overset{g_*}{\to} H_i(X') \overset{D^{-1}}{\to} H^{N-i}(S^N, S^N \setminus X') \overset{\delta^*}{\to} H^{N-i-1}(S^N \setminus X') = H^{N-i-1}(X^*)$$

is S-duality. Since Sq^r commutes with δ^* and is natural, $g_*^{-1}D^{-1}\mathrm{Sq}^r D g_* = S^{-1}\mathrm{Sq}^r S$. Q.E.D.

The Thom operations yield obstructions to immersions and embeddings up to homotopy type:

PROPOSITION 1.3 (THOM). *Let $(X, A) \overset{g}{\simeq} (X', A')$ be a homotopy equivalence and let i: $X' \to V^N$ be an immersion. Let φ be the composite $X \to X' \to V \overset{\psi}{\to}$* BPL, *where ψ classifies the stable tangent bundle of V. If $2 \neq 0$ in R assume also that $\pi_1(X) \to \pi_1(\mathrm{BPL})$ is zero. Then $\theta_r^R(X, A; \varphi)$ on $H_n(X, A; R)$ is zero if $r > N - n$. Suppose in addition that V is R-acyclic (e.g. $V = \mathbf{R}^N$), $A = \varnothing$ and i is an embedding. Then $\theta_r^R(X; \varphi)$ on $H_n(X; R)$ is zero if $r \geqslant N - n$.*

PROOF. We may assume i is an embedding, since an immersion $X \to V$ may be factored first as an embedding $X \to V'$ and then as an immersion $V' \to V$, where V' has the same dimension as V. Note also that the stable tangent bundle of V' is classified by $V' \to V \overset{\psi}{\to}$ BPL. For an embedding, the assertion follows directly from the definitions, since Sq^r is zero on $H^{N-n}(\ ; R)$ *if* $r > N - n$.

In case V is acyclic and $A = \varnothing$, then $H^{N-n}(V, V \setminus X) \overset{\delta^*}{\simeq} H^{N-n-1}(V \setminus X)$ and we gain one dimension as Sq^r commutes with δ^*.

2. Immersions of pseudomanifolds. In this section, we assume familiarity with the intersection homology theory of M. Goresky and R. MacPherson [**GM1**, **GM2**]. X^n will denote a compact stratified pl pseudomanifold, p a perversity, and $I_pH_*(X)$ the intersection homology groups.

By the Thom theorem, if X^n immerses in V^N, the operations $\theta_r^R(X; \varphi)$ are zero on $H_j(X; R)$ if $r > N - j$. We study the limit case $r = N - j$ and prove

THEOREM 2.1. *Under the hypothesis of Proposition* 1.3, *for all perversities p, there are maps*

$$I_p\theta_{N-j}^R(X; \varphi)\colon I_pH_j(X; R) \to I_pH_{2j-N}(X; R_\varepsilon)$$

which are natural with respect to changes of perversity and with respect to the map to homology. In particular, the composite map

$$I_pH_j(X;R) \to H_j(X;R) \overset{\theta^R_{N-j}(X;\varphi)}{\to} H_{2j-N}(X;R_\varepsilon)$$

is in the image of $I_pH_{2j-N}(X;R_\varepsilon) \to H_{2j-N}(X;R_\varepsilon)$.

Since the fundamental class $[X]$ of an R-orientable pseudomanifold X^n lies in $I_pH_n(X)$ for all perversities p, and in particular for $p=0$, we have the following:

COROLLARY 2.2. *Let X^n be an R-orientable pseudomanifold. Suppose X immerses up to homotopy type in $\mathbf{R}^m$. Then $\theta^R_{m-n}[X]$ is in the image of the map*

$$I_0H_{2n-m}(X;R_\varepsilon) \to H_{2n-m}(X;R_\varepsilon).$$

In particular, if X is normal, there is an $\alpha \in H^{m-n}(X;R)$ such that $\theta^R_{m-n}[X]$ is the image of $\alpha \cap [X]$ under the map

$$H_{2n-m}(X;R) \to H_{2n-m}(X;R_\varepsilon).$$

For orientable manifolds, $\cap[X]$ is the Poincaré duality isomorphism and one has the formula $\theta^{\mathbf{Z}}_{m-n}[X] = \overline{W}^{m-n} \cap [X]$, where $\overline{W}^{m-n}$ is the normal Stiefel-Whitney class. Thus for manifolds, no new nonimmersion results are obtained. On the other hand, for pseudomanifolds, the homomorphism $\cap[X]$ is not a surjection in general. For example, if X has the homotopy type of a suspension, it is the zero homomorphism. On the other hand, θ_r commutes with suspension. This shows the following

COROLLARY 2.3. *Let Y^{n-k} be an R-orientable pseudomanifold. Assume $\theta_r^R[Y] \neq 0$ (e.g., Y a manifold and $\overline{W}^r \neq 0$). Then $X = Y \wedge S^k$ does not immerse in $\mathbf{R}^N$ up to homotopy type for $N \leqslant n - r$.*

REMARK. In case $k=1$, this says ΣY does not immerse in $\mathbf{R}^N$. This can be seen directly as follows: By the Thom theorem, Y does not embed in $\mathbf{R}^{N-1}$ if $N-1 \leqslant n - 1 + r$. So X cannot immerse in $\mathbf{R}^N$ for $N \leqslant n + r$ since any immersion would embed the link Y in S^{N-1}.

The following class of examples should convince the reader that one can find many examples of nonimmersions for pseudomanifolds, which do not come from Thom's theorem.

COROLLARY 2.4. *Let W^n be a manifold satisfying $\overline{W}^{m-n} \neq 0$. Let $X \subset W$ be any subcomplex with $H^{m-n}(W,X;R) = 0$ and $H_{2n-m}(W;R_\varepsilon) \to H_{2n-m}(W,X;R_\varepsilon)$ injective. Then W/X does not immerse up to homotopy type in $\mathbf{R}^m$.*

For example, if $n > 2/3m$ we may take X to be any subcomplex of W which contains the $m-n$ skeleton of W and is contained in the $2n-m-1$ skeleton.

PROOF OF COROLLARY 2.4. Let $p\colon W \to W/X$ denote the projection. Then $\theta_{m-n}[W/X] = \theta_{m-n}(p_*[W]) = p_*\theta_{m-n}[W] = p_*(\overline{W}^{m-n} \cap [W]) \neq 0$, since θ is natural, p_* is injective in dimension $2n-m$, and $\overline{W}^{m-n} \neq 0$. On the other hand, since $H^{m-n}(W/X;R) = 0$, $\theta_{m-n}[W/X]$ is not the image of a cohomology class.

3. Proof of Theorem 2.1. By the proof of Proposition 1.3 we may assume i is an embedding and V is a compact R-oriented neighborhood of X'. The diagram

$$\begin{array}{ccc} H_j(X;R) & \xrightarrow{\theta_r^R(X;\varphi)} & H_{j-r}(X;R_\varepsilon) \\ \downarrow \wr f_* & & \downarrow \wr f_* \\ H_j(V;R) & & H_{j-r}(V) \\ \uparrow \wr D^{-1} & & \uparrow \wr D^{-1} \\ H^{N-j}(V,\partial V;R) & \xrightarrow{\mathrm{Sq}^r} & H^{N+r-j}(V,\partial V;R_\varepsilon) \end{array}$$

commutes, where D is the inverse of the Poincaré duality. For $r = N - j$, $\mathrm{Sq}^r x = x \cup x$ for degree $x = N - j$. Thus $f_*\theta_r^R(X;\varphi)(a) = (f_*(a) \cdot f_*(a))$ for $a \in H_j(X;R)$, where $\cdot$ is the intersection pairing in V.

Now the intersection product $\xi \cdot \eta$ may be defined as follows:

$$\xi \cdot \eta = \pi_*(\mu \cap \xi \times \eta),$$

where $\mu \in H^N(V \times V)$ is the image of the class in $H^N(V \times V, V \times V \setminus \Delta)$ dual to the diagonal Δ, and $\pi\colon V \times V \to V$ is projection onto the first factor. It follows that

$$f_*\xi \cdot f_*\xi = \pi_*(\mu \cap f_*\xi \times f_*\xi) = f_*\tilde{\pi}_*\big((f \times \mathrm{id})^*\mu \cap \xi \times f_*\xi\big),$$

where $\tilde{\pi}\colon X \times V \to X$.

Thus for $\xi \in H_j(X;R)$ we have the formula

$$f_*\theta_r^R(X;\varphi)(\xi) = f_*\tilde{\pi}_*\big((f \times \mathrm{id})^*\mu \cap \xi \times f_*\xi\big)$$

and hence $\theta_r^R(X;\varphi)(\xi) = \tilde{\pi}_*((f \times \mathrm{id})^*\mu \cap \xi \times f_*\xi)$ since f_* is an isomorphism.

Now if ξ is of perversity p in X (i.e. $\xi \in I_pH_*(X)$), $\xi \times f_*\xi$ is of perversity p in $X \times V$. Hence $\gamma = (f \times \mathrm{id})^*\mu \cap \xi \times f_*\xi$ is also of perversity p in $X \times V$. Since $\tilde{\pi}\colon X \times V \to X$ is a normally nonsingular projection, $\tilde{\pi}_*(\gamma)$ is of perversity p in X.

Thus we may define the map $I_p\theta_r^R(X;\varphi)$ by the formula

$$\xi \to \tilde{\pi}_*\big((f \times \mathrm{id})^*\mu \cap \xi \times f_*\xi\big),$$

where all maps are considered to be in intersection homology. The naturality properties of these maps are immediate from naturality properties of intersection homology maps.

REMARK. It is intriguing that we have only used that X immerses up to homotopy type. Can one find stronger results if one requires an actual immersion?

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