PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 96, Number 4, April 1986

THE HOMOTOPY TYPE OF CERTAIN LAMINATED MANIFOLDS

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ABSTRACT. Let M denote a connected $(n+1)$-manifold $(n \geq 5)$. A lamination G of M is an usc decomposition of M into closed connected n-manifolds. Daverman has shown that the decomposition space M/G is homeomorphic to a 1-manifold possibly with boundary. If $M/G = R^1$, we prove that M has the homotopy type of an n-manifold if and only if $\Pi_1(M)$ is finitely presented. In the case that $M/G = S^1$ we use the above result to construct an approximate fibration $f\colon M \to S^1$. We then discuss the important interactions of this study with that of perfect subgroups of finitely presented groups.

0. Introduction. Let M be an $(n+1)$-manifold. A *lamination* of M is an usc decomposition G of M into closed connected n-manifolds. The decomposition space M/G is always a 1-manifold, possibly with boundary [**4**, Theorem 3.3]. The main theorem concerns the case $M/G = R^1$. We show that M is homotopy equivalent to a closed codimension one submanifold if and only if its fundamental group $\Pi_1(M)$ is finitely presented. Finally, we apply this theorem to the case where $M/G = S^1$ to find an approximate fibration $f\colon M \to S^1$. As a corollary, if the Whitehead group $\mathrm{Wh}(\Pi_1(M))$ is trivial, then M is a locally trivial fibration over S^1.

The roots of this research spring from a more general investigation of decompositions of manifolds into closed connected manifolds of arbitrary fixed codimension. A detailed discussion of this investigation appears in a survey article by Daverman [**5**].

1. Statement of results. We state the main theorem and prove its corollaries. In this section M will always denote an $(n+1)$-manifold with $n \geq 5$, G an usc decomposition of M into closed connected n-manifolds, and $p\colon M \to M/G$ the decomposition map.

THEOREM 1.1. *Suppose M admits a lamination G with $M/G = R^1$. Then M admits another lamination G' with a locally flat element $g' \in G'$ such that the inclusion $i\colon g' \to M$ is a homotopy equivalence if and only if $\Pi_1(M)$ is finitely presented.*

This theorem gives information about other laminated manifolds.

COROLLARY 1.2. *Suppose M admits a lamination G with $M/G = S^1$. Then there is an approximate fibration $f\colon M \to S^1$.*

Received by the editors February 15, 1985. Presented at the 90th Annual Meeting of the Society, Louisville, Kentucky, on January 27, 1984.
1980 *Mathematics Subject Classification*. Primary 57N15; Secondary 55P15.
Key words and phrases. Lamination, decomposition, approximate fibration, homotopy type, finitely presented group, perfect subgroup.

COROLLARY 1.3. *Suppose M admits a lamination G with $M/G = S^1$ and* $\mathrm{Wh}(\Pi_1(M)) = 0$. *Then M is a locally trivial fibration over S^1.*

PROOF OF COROLLARIES 1.2 AND 1.3. Choose $g \in G$ locally flat [**4**, Lemma 6.1]. Then the complement of g, $M - g$, inherits the lamination $G - g$ from G. Since g is locally flat, the manifold $M - g$ deformation retracts to a compact manifold (in particular, a compact ENR). As a result $M - g$ is dominated by a finite CW-complex X. The group $\Pi_1(M - g)$ is therefore a retract of the finitely presented group $\Pi_1(X)$ and as such is itself finitely presented [**17**, Lemma 1.3]. Also the decomposition space $(M - g)/(G - g)$ is homeomorphic to R^1. Using Theorem 1.1 relaminate $M - g$ with a lamination G' having an element $g' \in G'$ with $i'\colon g' \to M - g$ a homotopy equivalence. Now $G' \cup \{g\}$ is a lamination of M [**4**, Theorem 3.4]. Let α: $g' \times [-1, 1] \to M$ be a bicollar on g'. Then $M' = M - \alpha(g' \times (0, 1/2))$ is an h-cobordism. There is a homeomorphism h: $M' - \alpha(g' \times 1/2) \to g' \times [0, 1)$ with $h(\alpha(z, 0)) = (z, 0)$ [**14**, Corollary 4.13.1]. Let $G'' = \{\alpha(g', s), h^{-1}(g' \times t), 0 < s \leq 1/2, 0 \leq t < 1\}$. Then G'' is a lamination of M [**4**, Theorem 3.4]. Finally p'': $M \to M/G'' = S^1$ is an approximate fibration [**9**, Theorem C]. This proves Corollary 1.2. If $\mathrm{Wh}(\Pi_1(M))$ vanishes, then p'' is homotopic to a locally trivial fibration [**9**, Corollary F].

2. Proof of the Main Theorem. We introduce the following notation. Suppose M admits a lamination G with $M/G = R^1$. Consequently, M has precisely two ends. Let N be any closed n-manifold which separates those ends. We denote by N^+ and N^- the closures of the components of $M - N$. One implication in the Main Theorem is easy since a compact manifold has finitely presented fundamental group (see the proofs of 1.2 and 1.3 above). For the other we suppose $\Pi_1(M)$ is finitely presented. Assume M has a PL triangulation.

Step I. Since $\Pi_1(M)$ is finitely presented, we can find a finite 2-complex P in M with inclusion inducing an isomorphism on fundamental groups. Choose $g \in G$ locally flat [**4**, Lemma 6.1] with $g \cap P = \emptyset$ and $P \subset g^+$. Let i: $g \to g^+$ and j: $g^+ \to M$ be inclusions. The induced homomorphism $i_\#$: $\Pi_1(g) \to \Pi_1(g^+)$ is a surjection [**6**, Lemma 3.1]. Since P carries $\Pi_1(M)$, $j_\#$: $\Pi_1(g^+) \to \Pi_1(M)$ is a surjection. Let l be a loop in g^+ representing an element of kernel$(j_\#)$. Since $i_\#$ is surjective, l is freely homotopic in g^+ to a loop l' in P. By construction, l' must bound a disk in P. Thus, kernel$(j_\#) = 1$ and $j_\#$ is an isomorphism.

Step II. Since $\Pi_1(g^+)$ and $\Pi_1(g)$ are both finitely presented, the kernel of $i_\#$: $\Pi_1(g) \to \Pi_1(g^+)$ is the normal closure of a finite set S of cardinality, say, m [**15**, Lemma 3.11]. Now $n \geq 5$ so we can attach m mutually disjoint, PL embedded 2-handles in g^+ to g with the cores of the attaching regions corresponding to representatives of the set S. We denote these handles by h_i^2, $1 \leq i \leq m$. These surgeries yield a new manifold pair (N_1^+, N_1) with N_1 the boundary of N_1^+ and the inclusion inducing an isomorphism of fundamental groups.

Step III. Denote the universal cover of M by $\tilde{M}$. For any subset U of M, $\tilde{U}$ will denote the preimage of U in $\tilde{M}$.

By pulling back the lamination of M, $\tilde{M}$ has a partition $\{\tilde{g} | g \in G\}$ which may fail to be a lamination only because it may not be usc. However, it is still the case that for any $\tilde{g}$, $H_*(\widetilde{g^+}, \tilde{g}) = 0$ [**4**, Lemma 6.2].

Let $h^2 = \bigcup_{i=1}^m h_i^2$. Consider the triple $(\widetilde{g^+}, \widetilde{g \cup h^2}, \tilde{g})$. Note that $\widetilde{g^+}$ and $\widetilde{g \cup h^2}$ are the universal covers of g^+ and $g \cup h^2$ respectively. The exact homology sequence of this triple follows:

$$\to H_k(\widetilde{g \cup h^2}, \tilde{g}) \to H_k(\widetilde{g^+}, \tilde{g}) \to H_k(\widetilde{g^+}, \widetilde{g \cup h^2}) \to .$$

By the above, $H_*(\widetilde{g^+}, \tilde{g}) = 0$. Thus, the boundary-induced homomorphism ∂_*: $H_{k+1}(\widetilde{g^+}, \widetilde{g \cup h^2}) \to H_k(\widetilde{g \cup h^2}, \tilde{g})$ is an isomorphism. By excision, $H_{k+1}(\widetilde{g^+}, \widetilde{g \cup h^2}) = H_{k+1}(\widetilde{N_1^+}, \widetilde{N_1})$. Each 2-handle h_i^2 is isomorphic to $D_i^2 \times D_i^{n-1}$ with $g \cap h_i^2 = S_i^1 \times D_i^{n-1}$.

Again by excision,

$$\begin{aligned} H_*(\widetilde{g \cup h^2}, \tilde{g}) &= H_*\left(\widetilde{\bigcup_{i=1}^m D^2 \times D^{n-1}}, \widetilde{\bigcup_{i=1}^m S^1 \times D^{n-1}}\right) \\ &= \bigoplus_{i=1}^m H_*(\widetilde{D_i^2 \times D_i^{n-1}}, \widetilde{S_i^1 \times D_i^{n-1}}). \end{aligned}$$

Therefore viewed as a $Z\Pi_1$-module

$$H_k(\widetilde{g \cup h^2}, \tilde{g}) = \begin{cases} \bigoplus_{i=1}^m Z\Pi_1, & k = 2, \\ 0, & k \neq 2. \end{cases}$$

Step IV. From Step III we have that

$$H_k(\widetilde{N_1^+}, \widetilde{N_1}) = \begin{cases} \bigoplus_{i=1}^m Z\Pi_1, & k = 3, \\ 0, & k \neq 3. \end{cases}$$

Both $\widetilde{N_1^+}$ and $\widetilde{N_1}$ are simply connected. Thus, the Hurewicz Theorem [**14**, p. 399] yields:

$$\Pi_k(\widetilde{N_1^+}, \widetilde{N_1}) = \begin{cases} \bigoplus_{i=1}^m Z\Pi_1, & k = 3, \\ 0, & k < 3. \end{cases}$$

Also $\Pi_k(\widetilde{N_1^+}, \widetilde{N_1}) = \Pi_k(N_1^+, N_1)$. So for each i, $1 \leq i \leq m$, we may choose a representative of a generator f_i: $(D^3, S^2) \to (N_1^+, N_1)$.

If $\dim(M) = n + 1 \geq 7$ we may assume that these are mutually disjoint PL embeddings (by general position). For the case $n = 5$ see the Appendix. For each i, $1 \leq i \leq m$, we attach a 3-handle h_i^3 with $f_i(D^3)$ as its core. We may assume that this collection of 3-handles is mutually disjoint. These surgeries yield a new manifold pair (N_2^+, N_2) with N_2 the boundary of N_2^+.

Step V. Let $h^3 = \bigcup_{i=1}^m h_i^3$. Consider the exact homology sequence of the triple $(\widetilde{N_1^+}, \widetilde{N_1 \cup h^3}, \widetilde{N_1})$:

$$\to H_k(\widetilde{N_1 \cup h^3}, \widetilde{N_1}) \to H_k(\widetilde{N_1^+}, \widetilde{N_1}) \to H_k(\widetilde{N_1^+}, \widetilde{N_1 \cup h^3}) \to .$$

By excision, $H_*(\widetilde{N_1^+}, \widetilde{N_1 \cup h^3}) = H_*(\widetilde{N_2^+}, \widetilde{N_2})$. By construction

$$i_*\colon H_k(\widetilde{N_1 \cup h^3}, \widetilde{N_1}) \to H_k(\widetilde{N_1^+}, \widetilde{N_1})$$

is an isomorphism for all k. Thus, $H_*(\widetilde{N_2^+}, \widetilde{N_2}) = 0$ so $\Pi_*(\widetilde{N_2^+}, \widetilde{N_2}) = 0$ and $\Pi_*(N_2^+, N_2) = 0$.

Step VI. A similar argument can be given to show that $\Pi_*(N_2^-, N_2) = 0$. However, this follows from the fact that the cobordism bounded by g and N_2 is precisely that obtained from the plus construction of Quillen [**12**]. As a result, the inclusion $i\colon N_2 \to M$ is a homotopy equivalence.

Step VII. Finally we construct a new lamination G' of M with $N_2 \in G'$ [**6**, Theorem 4.2]. Let $g' = N_2$.

This completes the proof of Theorem 1.1 in the case that M is PL. However, since $n \geq 5$ this argument can be generalized to topological manifolds [**11**, Essay III].

3. The fundamental group of a laminated manifold M. In light of Theorem 1.1 we pose the following question:

Question 3.1. Suppose M admits a lamination G with $M/G = R^1$. Must $\Pi_1(M)$ be finitely presented?

Suppose the answer is negative. Let $g_t = p^{-1}(t) \subset M$ for $t \in R^1$. Since $i\colon g_0 \to M$ induces a surjection on fundamental groups [**6**, Lemma 3.1] and the kernel of this induced homomorphism is perfect [**7**, Lemma 2.4], then $\Pi_1(g_0)$ would be a finitely presented group with a perfect normal subgroup which is not the normal closure of a finite set. Question 3.1 is equivalent to an apparently quite difficult problem in combinatorial and homological group theory.

DEFINITION 3.2. A group G is *almost finitely presented* if there exists a short exact sequence $0 \to R \to F \to G \to 0$ with F finitely generated and free and R/R' finitely generated as a ZG module (R' is the commutator subgroup of R).

Question 3.3. Is every almost finitely presented group G finitely presented [**18**, Problem F10, p. 387]?

The existence of a laminated M with a nonfinitely presented fundamental group is slightly more restrictive. Denote by N_t the closure of the bounded component of $M - \{g_0 \cup g_t\}$. Then the inclusion of g_0 into N_t induces a homomorphism of finitely presented groups with perfect kernel for $t > 0$. The existence of such a laminated M would imply a positive answer to the following question.

Question 3.4. Does there exist a finitely presented group G with perfect normal subgroup P such that

(1) $P = \bigcup_{i=1}^{\infty} P_i$,
(2) $P_i \subset P_{i+1}$ for all i,
(3) each P_i is perfect and is the normal closure of a finite set, and
(4) P is not the normal closure of a finite set?

In each of the laminated manifolds constructed by us the kernel P_i is actually the normal closure of a finitely generated perfect group [**7**]. Thus, we also ask

Question 3.5. Does there exist a finitely presented group G with perfect normal subgroup P such that

(1) same as in 3.4,
(2) same as in 3.4,
(3) each P_i is the normal closure in G of a finitely generated perfect group,
(4) same as in 3.4?

Appendix. We complete Step IV of the proof of Theorem 1.1 for the case $n = 5$. Recall that $\Pi_3(N_1^+, N_1)$ is a free $Z\Pi_1$ module on m generators $\{f_i\colon (D^3, S^2) \to (N_1^+, N_1) | 1 \leq i \leq m\}$. Since the dimensions of D^3 and N_1 are 3 and 6 respectively and $\Pi_k(N_1^+, N_1) = 0$ for $k < 3$, we may assume that the f_i's are PL embeddings [**8**, Corollary 8.2]. However, they may not be mutually disjoint. We show how to remove the intersections using the Whitney Lemma [**13**, Lemma 5.12].

For each i denote $(f_i(D^3), f_i(S^2))$ by (D_i^3, S_i^2). By general position we may assume that any pair of these 3-disks intersect in their interiors in at most finitely many points distinct from the intersection points of any other pair.

Suppose $x \in (D_1^3 \cap D_2^3) \subset (N_1^+ - N_1)$. We show how to eliminate this intersection without disturbing any other existing intersection points or introducing any new intersection points. The result then follows by induction.

Choose PL arcs α_1 and α_2 joining x to S_1^2 and S_2^2 respectively. We may assume α_i misses other intersection points and $\text{int}(\alpha_i) \cap \{S_i^2 \cup x\} = \{x_i \cup x\}$, where $x_i \in S_i^2$. Since the inclusion of N_1 into N_1^+ induces an isomorphism on fundamental groups, there is an arc β joining x_1 and x_2 in N_1 so that the simple closed curve $\gamma = \alpha_2 \alpha_1^{-1} \beta$ is trivial in N_1^+ and $\gamma \cap S_j^2 = \varnothing$ for $j > 2$. Let B be a PL 2-disk bounded by γ with $B \cap \{N_1 \cup D_1^3 \cup D_2^3\} = \gamma$ and $B \cap D_j^3 = \varnothing$ for $j > 2$.

For the following we use [**13**, Chapter 5] as a general reference. Let (L, L_1, L_2) be a regular neighborhood of B in (N_1^+, D_1^3, D_2^3). Then each (L, L_i) is an unknotted ball pair. However, $\text{bdy}(L_1)$ and $\text{bdy}(L_2)$ are linked in $\text{bdy}(L)$. For each i, $\text{bdy}(L_i) \cap N_1 = E_i$, a PL 2-ball. Also β joins $\text{int}(E_1)$ to $\text{int}(E_2)$.

By an isotopy of N_1 with support in an arbitrary neighborhood of β we "push $\text{int}(E_1)$ through $\text{int}(E_2)$ along β" with E_1 and E_2 oriented so as to unlink $\text{bdy}(L_1)$ and $\text{bdy}(L_2)$. We extend this isotopy to N_1^+ using a small collar on N_1.

The effect of this isotopy is to introduce an intersection x' of D_1^3 and D_2^3 with $\epsilon(x') = -\epsilon(x)$, where $\epsilon(x) \in Z\Pi_1$ is the standardly defined intersection number. Now the Whitney Lemma applies to remove both x and x' as intersections.

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