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# A TRIVIAL LINK WITH NO LINEAR UNLINKING

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ABSTRACT. Not every relative link (finite collection of disjoint polygonal spanning arcs of a cube) that is trivial allows a linear unlinking.

**I. Introduction.** The following linearization question is from [**1**, Question 9]:

I.1 *Question.* Given a polygonal unknotted spanning arc $A$ of a cube $C$, is there an unknotting that keeps the endpoints of $A$ fixed and is *linear* at each stage?

In this paper we give a negative answer to the following related question:

I.2 *Question.* Given disjoint polygonal spanning arcs $E_1, E_2, \ldots, E_n$ in a cube $C$ and an isotopy of the set $E_1 \cup E_2 \cup \cdots \cup E_n$ in $C$ keeping the endpoints fixed and taking each $E_i$ onto a line segment, is there such an isotopy that is linear at each stage?

There is an example in [**2**] of two isotopic imbeddings of a 1-complex in $E^3$ for which there is no linear isotopy. Our example is an improvement in that

(1) each component of our example is an arc,

(2) our imbedding is trivial, and

(3) the example and techniques of this paper will be used in a sequel to answer Question I.1.

**II. The construction.** Let $C = \{(x,y,z)/ -15.0003 \leq x \leq 15.0003,\ -3 \leq y \leq 3,\ 0 \leq z \leq 801\}$. $C$ is not a cube, but it can be changed into a cube by a linear homeomorphism. Let $E_1$ and $E_2$ be as in Figure 1, where they are shown as fattened arcs so that the linking of the other arcs in Figure 1 can be more easily seen. More exactly,

$$E_1 = [(-5.00625, -1, 0), (0, -1, 801)] \cup [(0, -1, 801), (5.00625, -1, 0)],$$
$$E_2 = [(-5.1, 1, 0), (0, 1, 51)] \cup [(0, 1, 51), (5.1, 1, 0)].$$

Then, for $1 \leq z \leq 1.00026$, let $A(z)$ be an arc consisting of three line segments and containing as a subset

$$[(-4.997, 3, z), (-5.0001, -1, z)] \cup [(4.997, 3, z), (5.0001, -1, z)].$$

Let $B(z)$ be an arc consisting of three line segments and containing

$$[(-4.997, -3, z), (-5.0001, 1, z)] \cup [(4.997, -3, z), (5.0001, 1, z)].$$

Let $C(z)$ be an arc consisting of three line segments and containing

$$[(-15.0003, -3, z), (5.0001, 1, z)] \cup [(15.0003, -3, z), (-5.0001, 1, z + .000001)].$$

Finally, let $D(z)$ be an arc consisting of three line segments and containing

$$[(-15.0003, 3, z), (5.0001, -1, z - .000001)] \cup [(15.0003, 3, z), (-5.0001, -1, z)].$$

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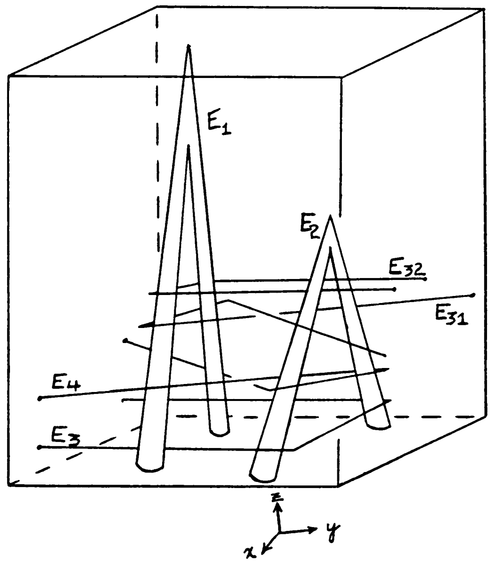


FIGURE 1

Next let $E_3 = B(1)$, $E_4 = C(1.000001)$, $E_5 = C(1.000003)$; for $i = 6, 8, 10, \ldots,$ 28 let $E_i = A(1+(i-5)\times 10^{-5})$; for $i = 7, 9, 11, \ldots, 29$ let $E_i = B(1+(i-5)\times 10^{-5})$; let $E_{30} = D(1.000247), E_{31} = D(1.000249)$ and $E_{32} = A(1.00026)$.

The collection $\{E_1, E_2, \ldots, E_{32}\}$ is a trivial link. The unlinking can be accomplished by unlinking $E_{32}$ from $E_1$ and $E_2$, then unlinking $E_{31}$ from $E_1$ and $E_2$, then $E_{30}, E_{29}$, etc.

We prove that no unlinking is linear by demonstrating a limited range of vertical motion for the interior vertex of $E_1$ and for that of $E_2$. Within this limited range of motion, $E_{32}$ cannot be unlinked from $E_1$ and $E_2$.

**III. Limitations on the motion of the link.** Let the interior vertex of $E_1$ and that of $E_2$ be given by $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$, respectively. Each arc $E_i$ has a leg $E_i^1$, whose point of attachment to $\operatorname{Bd} C$ has a negative $x$-coordinate, a leg $E_i^2$, whose point of attachment has a positive $x$-coordinate, and, for $i \neq 1, 2$, a crosspiece.

In §II, each arc $E_i$ had specific coordinates. In this section we use the same symbol $E_i$ to refer to the image of arc $E_i$ at a stage of an isotopy that is linear at each stage.

The $yz$-slope of a line segment in $C$ is defined to be the slope of its projection into the $yz$-plane.

In this section we assume

(1) $601 \leq z_1 \leq 801$ and $50 \leq z_2 \leq 53$.

(2) $\max_{j=1,2} |yz\text{-slope } E_i^j| \leq 2$ for $i = 3, 4, 5, \ldots, 32$.

If these two conditions are maintained throughout an isotopy that is linear on the $E_i$'s at each stage, then $E_3, E_4, \ldots, E_{32}$ will remain linked with $E_1$ and $E_2$ as in Figure 1.

III.1 LEMMA. *Suppose $E_1, E_2$ and $E_{32}$ are constructed as in* §II, *and an isotopy is given that is linear at each stage on these three arcs and satisfies conditions* (1) *and* (2) *as they apply to $E_1, E_2$, and $E_{32}$. Then at each stage yz-slope $E_{32}^2 \neq -2$ or yz-slope $E_{32}^1 > .001$.*

PROOF. The proof consists of two parts:

(A) If $yz$-slope $E_{32}^2 = -2$, then $x_2 > 1.5$.

(B) If $yz$-slope $E_{32}^1 \leq .001$, then $x_2 < 1.5$.

*Part* A. It is sufficient to show that even the extreme case $yz$-slope $E_{32}^2 = -2$ and $x_2 = 1.5$ would force $E_{32}^2$ to cross the plane determined by $E_2$ outside of the triangle, two of whose sides are $E_2^1$ and $E_2^2$. Move the upper endpoint of $E_{32}^2$ along a ray parallel to the positive $x$-axis until $E_{32}^2$ intersects $E_2^2$ at the point $(u_2, v_2, w_2)$.

A.1 *An upper bound for $u_2$.* Projecting into the $yz$-plane, we see that a lower bound for $w_2$ occurs when $(x_2, y_2, z_2) = (1.5, 3, 50)$. Find the intersection of the $yz$-projections of $E_{32}^2$ and $E_2^2$: $w_2 \geq 4.62$. Then, project into the $xz$-plane, take $(x_2, z_2) = (1.5, 53)$, and use $w_2 \geq 4.62$: $u_2 \leq 4.79$.

Next, move the upper endpoint of $E_{32}^2$ along a ray parallel to the negative $x$-axis until $E_{32}^2$ intersects $E_1^2$ at the point $(u_1, v_1, w_1)$. (If $E_{32}^2$ is not long enough to intersect $E_1^2$, then the movement is until the *line* determined by $E_{32}^2$ intersects $E_1^2$.)

A.2 *A lower bound for $u_1$.* Projecting into the $yz$-plane, we see that an upper bound for $w_1$ occurs when $(y_1, z_1) = (-3, 601)$. Find the intersection of the $yz$-projections of $E_1^2$ and $E_{32}^2$: $w_2 \leq 9.07$. Project into the $xz$-plane, take $(x_1, z_1) = (-15.0003, 601)$, and use $w_1 \leq 9.07$: $u_1 \geq 4.70$.

A.3 *An upper bound for $u_1$.* Use $(y_1, z_1) = (3, 601)$ and $(y_2, z_2) = (-3, 50)$ to find $v_1 \leq -.94$ and $v_2 \geq .52$. By an argument from similar triangles,

$$(4.9997 - u_1)/(4.9997 - u_2) \geq 3.94/2.48.$$

Then the estimate $u_2 \leq 4.79$ above implies $u_1 \leq 4.67$. This contradicts the earlier estimate $u_1 \geq 4.7$ and Part A is complete.

*Part* B. It is enough to show that the case $yz$-slope $E_{32}^1 = .001$ and $x_2 = 1.5$ is impossible. Move the free endpoint of $E_{32}^1$ along a ray parallel to the negative $x$-axis until $E_{32}^1$ intersects $E_2^1$ at the point $(u_2, v_2, w_2)$.

B.1 *A lower bound for $u_2$.* Projecting into the $yz$-plane, we see that a lower bound for $w_2$ occurs when $(x_2, y_2, z_2) = (1.5, -3, 50)$ and is $w_2 \geq .998$. In the $xz$-plane, use $w_2 \geq .998$ and $(x_2, z_2) = (1.5, 53)$: $u_2 \geq -4.98$.

Now move the free endpoint of $E_{32}^1$ along a ray parallel to the positive $x$-axis until the line containing $E_{32}^1$ intersects $E_1^1$ in the point $(u_1, v_1, w_1)$.

B.2 *An upper bound for $u_1$.* In the $yz$-plane, take $(y_1, z_1) = (3, 601)$ to find $w_1 \leq .997$. Then, in the $xz$-plane, use $(x_1, z_1) = (15.0003, 601)$ to find $u_1 \leq -4.97$.

B.3 *A lower bound for $u_1$.* Use $(y_1, z_1) = (3, 601)$ and $(y_2, z_2) = (-3, 50)$ to find

$$(u_1 + 4.9997)/(u_2 + 4.9997) \geq (3 + .9933)/(3 - .9201)$$

which, with the estimate $u_2 \geq -4.98$, implies $u_1 \geq -4.962$. This contradicts the estimate $u_1 \leq -4.97$. □

III.2 LEMMA. *Suppose $E_1, E_2$, and $E_3$ are constructed as in* §II, *and an isotopy is given that is linear at each stage on these three arcs and satisfies conditions*

(1) *and* (2) *as they apply to* $E_1, E_2$, *and* $E_3$. *At each stage, if* $z_1 = 601$, *then* $\max_{i=1,2} yz$-*slope* $E_3^i > .001$.

PROOF. Assume, temporarily, that $x_1 = x_2 = 0$, and that $y_1, y_2, z_2$ and $E_3^1$ are chosen to maximize $yz$-slope of $E_3^1$. This can be done, given the correct values of $y_1, y_2$, and $z_2$, by raising the free end of $E_3^1$ until $E_3^1$ touches $E_1^1$ and $E_2^1$. Then, due to the symmetry induced by $x_1 = x_2 = 0$, max $yz$-slope $E_3^2$ can be realized in the same way, *without* changing $y_1, y_2$, or $z_2$ or moving $E_3^1$. Now the $yz$-projection of $E_3$ is a single line segment. It is sufficient to prove that the case $z_1 = 601$ and $yz$-slope $E_3 = .001$ is impossible.

*An upper bound for* $d_1$. For $i = 1, 2$, $d_i$ is the distance between the two points of $E_i$ that project onto $(b_i, c_i) = (yz\text{-projection } E_3) \cap (yz\text{-projection } E_i)$. Use $(y_1, z_1) = (-3, 601)$ to find $c_1 \geq 1.0019$. Then from the $xz$-projection of $E_1$,

$$d_1/10.0125 \leq (601 - 1.0019)/601 \quad \text{or} \quad d_1 \leq 9.9959.$$

*A lower bound for* $d_2$. Use $(y_2, z_2) = (3, 50)$ to find $c_2 \leq 1.0041$ and $d_2 \geq 9.9951$.

*An upper bound for* $d_2$. Use $(y_1, z_1) = (3, 601)$ and $(y_2, z_2) = (-3, 50)$ to conclude

$$(9.9994 - d_2)/(9.9994 - d_1) \geq (-3 - .9196)/(-3 + .9933)$$

and $d_2 \leq 9.9926$. The two estimates for $d_2$ are contradictory.

What if the restriction $x_1 = x_2 = 0$ is relaxed? For given $y_1, y_2, z_1$, and $z_2$, moving $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ along lines parallel to the $x$-axis cannot reduce $\max_{i=1,2} yz$-slope $E_3^i$. To see this, starting with $x_1 = x_2 = 0$, change $x_2$ to any desired value, allowing $E_3$ to force movement of $(x_1, y_1, z_1)$ parallel to the $x$-axis. Then change $x_1$ to any value you wish. One leg of $E_3$ will be forced to increase in slope. □

III.3 LEMMA. *Suppose* $E_1, E_2$, *and* $E_{32}$ *are constructed as in* §II, *and an isotopy is given that is linear at each stage on these three arcs and satisfies conditions* (1) *and* (2) *as they apply to* $E_1, E_2$, *and* $E_{32}$. *At any stage, if* $z_2 = 50$, *then* $\max_{i=1,2} yz$-*slope* $E_{32}^i > .001$.

PROOF. As in Lemma III.2 it is sufficient to assume $x_1 = x_2 = 0$ and prove that the case $z_2 = 50$ and $yz$-slope $E_{32} = .001$ is impossible.

Use $(y_2, z_2) = (-3, 50)$ to find $c_2 \geq .9981$ (where $b_2, c_2$, and $d_2$ are defined as in Lemma III.2 using $E_{32}$ in place of $E_3$) and $d_2 \leq 9.9964$. Use $(y_1, z_1) = (3, 601)$ to find $c_1 \leq .9962$ and $d_1 \geq 9.9959$. From

$$(d_1 - 9.9994)/(d_2 - 9.9994) \geq (3 + .9933)/(3 - .9201)$$

it follows that $d_1 \leq 9.9937$. □

III.4 LEMMA. *Suppose* $E_1, E_2$, *and* $E_3$ *are constructed as in* §II, *and an isotopy is given that is linear at each stage on these three arcs and satisfies conditions* (1) *and* (2) *as they apply to* $E_1, E_2$, *and* $E_3$. *At any stage, if* $z_2 = 53$, *then* $\max_{i=1,2} yz$-*slope* $E_3^i > .001$.

PROOF. As for Lemma III.2 it is enough to assume $x_1 = x_2 = 0$ and prove that the case $z_2 = 53$ and $yz$-slope $E_3 = .001$ is impossible. Use $(y_1, z_1) = (-3, 601)$ to find $c_1 \geq 1.0019$; $z_1 = 801$ to find $d_1 \leq 10.0000$; and $(y_2, z_2) = (3, 53)$ to find $c_2 \leq 1.0041$ and $d_2 \geq 10.0067$. Finally, use $(y_1, z_1) = (-3, 601)$ and $(y_2, z_2) = (3, 53)$

to conclude that $(d_2 - 9.9994)/(d_1 - 9.9994) \leq (-3 - 1.0379)/(-3 + 1.0034)$ and $d_2 \leq 10.0006$. $\square$

Let $m_i = \min_{j=1,2} yz\text{-slope } E_i^j$ and $M_i = \max_{j=1,2} yz\text{-slope } E_i^j$.

III.5 LEMMA. *Suppose $E_1, E_2, E_i$ and $E_{k(i)}$ are constructed as in §II, where $i = 5, 6, 7, \ldots, 28, 29$, or 30, and*

$$k(i) = \begin{cases} 6 & \text{if } i = 5, \\ 3 & \text{if } i = 6, \\ i+1 & \text{if } i = 7, 9, 11, \ldots, 27, \\ i-1 & \text{if } i = 8, 10, 12, \ldots, 28, \\ 32 & \text{if } i = 29, \\ 29 & \text{if } i = 30, \end{cases}$$

*or $E_1, E_2, E_i$, and $E_{l(i)}$ are constructed as in §II where $i = 6, 7, 8, \ldots, 27, 28, 29$, and*

$$l(i) = \begin{cases} i+1 & \text{if } i = 6, 8, 10, \ldots, 28, \\ i-1 & \text{if } i = 7, 9, 11, \ldots, 29. \end{cases}$$

*Suppose also that an isotopy is given that is linear at each stage on $E_1, E_2, E_i$ and $E_{k(i)}$ (or on $E_1, E_2, E_i$, and $E_{l(i)}$) and satisfies conditions (1) and (2) as they apply to these four arcs. Then, at any stage, if $m_i \geq .00099$, then $M_{k(i)} \leq -1.4m_i$ (or if $M_i \leq -.001$, then $m_{l(i)} \geq -1.4M_i$).*

PROOF. For a fixed $(i, j, k)$ with $k = 1$ or 2, $j = 3, 5, 6, 7, \ldots, 28, 29, 30, 32$, and $i = 1$ if $j$ is odd, $i = 2$ if $j$ is even, let

$$(b, c) = (yz\text{-projection } E_i) \cap (yz\text{-projection } E_j^k).$$

The smallest value for $|b|$ occurs for $E_2$ and $E_{32}^1$ (or $E_{32}^2$) when $(y_2, z_2) = (-3, 50)$ and $yz\text{-slope } E_{32}^1 = -2$ (assumption 2 from the beginning of §III is still in effect). Then $|b| \geq .523$ and, for each $i$, each leg of $E_{k(i)}$ must pass above $E_i$ if $i$ is odd (or below it if $i$ is even) at a point at least 3.523 units from the $xz$-wall of attachment of $E_i$ and at most 2.477 units from the $xz$-wall of attachment of $E_{k(i)}$. Comparing heights at these crossover points we have

$$|M_{k(i)}| \geq (|m_i| 3.523 - .00001)/2.477$$

or

$$|M_{k(i)}/m_i| \geq 3.523/2.477 - .00001/(2.477)(.00099) \geq 1.4.$$

The same estimates hold for $m_{l(i)}$. $\square$

III.6 LEMMA. *Suppose $E_1, E_2, E_3, E_4, E_5$, and $E_6$ (or $E_1, E_2, E_{29}, E_{30}, E_{31}$, and $E_{32}$) are constructed as in §II, and an isotopy is given that is linear at each stage on these six arcs and satisfies conditions (1) and (2) as they apply to these six arcs. Then, at any stage, if $M_3 \geq .001$, then $M_6 \leq -1.3M_3$ (or if $M_{32} \geq .001$, then $M_{29} \leq -1.3M_{32}$).*

PROOF. Assume $M_{32} \geq .001$ and $M_{32} = yz\text{-slope } E_{32}^1$ (this will be seen to be harder than the case $yz\text{-slope } E_{32}^2 = M_{32}$). Let

$$(b, c) = (yz\text{-projection } E_{32}^1) \cap (yz\text{-projection } E_2).$$

Then $b \leq 1.04$, so that $E^1_{31}$ crosses under $E^1_{32}$ at a point with $y$-coordinate at most 1.04. A comparison of heights at this crossover point yields

$$\begin{aligned} yz\text{-slope } E^1_{31} &\geq yz\text{-slope } E^1_{32} - .000001/1.96 \\ &\geq .999\ yz\text{-slope } E^1_{32}. \end{aligned}$$

Likewise, $E^2_{30}$ crosses under $E^1_{31}$ so that $yz$-slope $E^2_{30} \geq .99$ ($yz$-slope $E^1_{32}$) and $E^1_{30}$ crosses under $E^2_{30}$ so that $yz$-slope $E^2_{30} \geq .99$ ($yz$-slope $E^1_{32}$). By Lemma III.5

$$\begin{aligned} M_{29} &\leq -1.4 m_{30} \leq -1.4(.99) yz\text{-slope } E^1_{32} \\ &\leq -1.3\ yz\text{-slope } E^1_{32}. \end{aligned}$$

The first half of the lemma is proven in the same manner. □

**IV. A negative answer to Question I.2.** Let $H\colon ((\bigcup_{i=1}^{32} E_i) \times [0,1]) \to C$ be a linear isotopy fixed on $\bigcup_{i=1}^{32} \operatorname{Bd} E_i$.

*Case* A. *At each stage of* $H$, $601 \leq z_1 \leq 801$ *and* $50 \leq z_2 \leq 53$. Then, by Lemma III.1, at each stage $yz$-slope $E^2_{32} \neq -2$ or $yz$-slope $E^1_{32} > .001$. But if $yz$-slope $E^1_{32} > .001$ then, by Lemma III.6, $M_{29} \leq -1.3 M_{32}$, and by Lemma III.5,

$$\begin{aligned} &m_{28} \geq -1.4 M_{29} \geq 1.3(1.4) M_{32}, \\ &M_{27} \leq -1.4 m_{28} \leq -1.3(1.4)^2 M_{32}, \\ &\quad\vdots \\ &m_6 \geq -1.4 M_7 \geq 1.3(1.4)^{23} M_{32}, \\ &M_3 \leq -1.4 m_6 \leq -1.3(1.4)^{24} M_{32} \leq -1.3(1.4)^{24}.001 < -2. \end{aligned}$$

Since $M_3 \leq -2$ is clearly impossible, it must be that $yz$-slope $E^2_{32} \neq -2$ at each stage of $H$ and, symmetrically, $yz$-slope $E^1_{32} \neq -2$. Then $H$ is not an unlinking.

*Case* B. *For some* $t$, *at stage* $H_t$, $z_1 = 601, z_2 = 50$, *or* $z_2 = 53$. Let $t_0$ be the smallest such $t$. If $z_1 = 601$ at stage $H_{t_0}$, by Lemma III.2, $M_3 > .001$. Lemma III.5 implies $M_6 \leq -1.3 M_3$. Successive applications of Lemma III.6 lead to $M_{32} \leq -1.3(1.4)^{24}.001 < -2$ which implies $yz$-slope $E^2_{32} = -2$ for some $t_1 \in (0, t_0)$, seen to be impossible in Case A.

If $z_2 = 50$, by Lemma III.3, $M_{32} > .001$ and using Lemmas III.5 and III.6, $M_3 < -2$, an impossibility.

If $z_2 = 53$, then $M_3 > .001$ and $M_{32} < -2$, seen to be impossible in Case A.

In summary, Cases A and B are exhaustive, Case B is vacuous, and a Case A isotopy is not an unlinking.

## References


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