THE DEPTH OF TRANCHES IN \(\lambda\)-DENDROIDS

LEE MOHLER¹

ABSTRACT. According to the well-known theory of Kuratowski, any hereditarily decomposable chainable continuum admits a decomposition into tranches. These tranches are themselves chainable and thus admit decompositions into their own tranches. We may thus define nested sequences \(\{T_\alpha\}\) of tranches-within-tranches, indexed by countable ordinals \(\alpha\), and finally terminating in a singleton set. E. S. Thomas, Jr. has asked whether, for a given continuum \(C\), there is a countable ordinal bound on the length of all such nests \(\{T_\alpha\}\) in \(C\). We answer Thomas’s question in the affirmative. By generalizing the definitions, we obtain the same result for \(\lambda\)-dendroids. We also answer, for chainable continua, a related question of Illiadis.

1. Introduction. According to the well-known theory of Kuratowski, any hereditarily decomposable chainable continuum admits a decomposition into tranches. (Technical terms will be defined below.) These tranches are themselves chainable continua and consequently admit decompositions into tranches. We may thus define nests \(\{T_\alpha\}\) of tranches-within-tranches, indexed by countable ordinals \(\alpha\). Any individual such nest must (by metrizability) end in a point for some countable ordinal \(\alpha\). In [8] E. S. Thomas, Jr. raised the question whether, for a given chainable continuum \(C\), there is a countable ordinal bound on the length of all nests of tranches-within-tranches in \(C\). In this paper we answer Thomas’s question in the affirmative (Corollary 2.9 below). Indeed, by modifying the definition slightly, we obtain the same result for the wider class of \(\lambda\)-dendroids (Corollary 2.8 below). The proof applies the “crookedness index” of Krasinkiewicz and Minc [6] to sequences of “half-tranches” obtained in Lemma 2.3 below, thus limiting their length. The author is indebted to Lex Oversteegen for suggesting the usefulness of [6] in attacking Thomas’s problem and for observing the generalizability of the author’s original proof from chainable continua to \(\lambda\)-dendroids. In §3 we apply the results of §2 to obtain an affirmative answer to a question of Illiadis [5] on nests of continua in hereditarily decomposable chainable continua. §4 is devoted to questions and examples.

A \(\lambda\)-dendroid is a hereditarily decomposable, hereditarily unicoherent (metric) continuum. Given a \(\lambda\)-dendroid \(X\) and points \(x, y \in X\), there is a unique subcontinuum \([x, y]\) of \(X\), irreducible with respect to containing both \(x\) and \(y\). By Kuratowski’s structure theory for irreducible continua (see [7, §48]; see also [8]), we have the following theorem.

Received by the editors October 18, 1984 and, in revised form, April 30, 1985.
1980 Mathematics Subject Classification. Primary 54F20; Secondary 54F50.
Key words and phrases. \(\lambda\)-dendroid, chainable, hereditarily decomposable, tranche.
¹The research presented here was supported by the program of scientific exchanges between the National Academy of Sciences (U.S.A.) and the Polish Academy of Sciences.
THEOREM 1.1. Let X be a \( \lambda \)-dendroid and let \( x, y \in X \) with \( x \neq y \). Then there is a monotone mapping \( f^* : [x, y] \to [0, 1] \) such that if \( f : [x, y] \to [0, 1] \) is any other monotone map, then there is a monotone map \( m : [0, 1] \to [0, 1] \) such that \( f = m \circ f^* \). Moreover, for every \( t \in [0, 1] \), \( f^{-1}(t) \) has void interior in \([x, y]\).

The continua \( f^{-1}(t) \) in the above theorem are called tranches of the continuum \([x, y]\). It is not difficult to verify that if \( f' : [x, y] \to [0, 1] \) is any other monotone map with every set \( f'^{-1}(t) \) having void interior in \([x, y]\), then the map \( m \) of 1.1 is a homeomorphism. Thus the decomposition of \([x, y]\) into tranches is unique.

Let \( X \) be a \( \lambda \)-dendroid. We may define a nested transfinite sequence of sub-continua of \( X \) as follows: Let \( T_0 = X \). For successor ordinals \( \alpha = \beta + 1 \), if \( T_\beta \) is nondegenerate, let \( x_\alpha \) and \( y_\alpha \) be distinct points of \( T_\beta \) and let \( T_\alpha \) be a tranche of \([x_\alpha, y_\alpha]\). For limit ordinals \( \alpha \), let \( T_\alpha = \bigcap_{\beta < \alpha} T_\beta \). Note that we also have \( T_\alpha = \bigcap_{\beta < \alpha} [x_\beta, y_\beta] \). As soon as some \( T_\alpha \) is a degenerate continuum, we stop the sequence. Since \( X \) is metrizable, any given sequence like the above must stop at some countable ordinal \( \alpha \). We will show below that in fact there is a countable ordinal \( \tau(X) \) such that every sequence of tranches-within-tranches defined as above in \( X \), stops at or before \( \tau(X) \). In what follows we will assume some familiarity with Kuratowski’s structure theory for irreducible continua. (See [7, §48] or [8].)

2. Sequences of tranches and half-tranches.

DEFINITION 2.1. Let \( X \) be a \( \lambda \)-dendroid, \( x, y \in X \) with \( x \neq y \) and let \( f^* : [x, y] \to [0, 1] \) be as in Theorem 1.1. Let \( t_0 \in [0, 1] \) and \( T = f^{-1}(t_0) \) be the associated tranche of \([x, y]\). Let \( L = T \cap \text{cl}(f^{-1}([0, t_0])) \) and \( R = T \cap \text{cl}(f^{-1}((t_0, 1])) \). (If \( t_0 = 0 \) or \( 1 \), one of these sets will be empty.) Then \( T = L \cup R \). \( L \) and \( R \) will be called halves of \( T \) and half-tranches of \([x, y]\).

The proof of the following lemma is straightforward.

LEMMA 2.2. Let \( X \) be a \( \lambda \)-dendroid, \( x, y \in X \) with \( x \neq y \) and let \( L \) and \( R \) be the two halves of the tranche \( T \) of \([x, y]\). Then if \( z \) is any point of \([x, y]\), \( L \) is a half-tranche of either \([x, z]\) or \([z, y]\). Similarly for \( R \).

LEMMA 2.3. Let \([x_1, y_1] \supset T_1 \supset [x_2, y_2] \supset \cdots \) be a transfinite sequence defined as in the last paragraph of §1. Suppose that for some countable ordinal \( \alpha_0 \), we have \( T_\alpha \) nondegenerate for each \( \alpha < \alpha_0 \). Then there is a sequence \([z_1, w_1] \supset H_1 \supset [z_2, w_2] \supset H_2 \supset \cdots \) with \( H_\alpha \) nondegenerate for each \( \alpha < \alpha_0 \) and \( H_\alpha = \bigcap_{\beta < \alpha} H_\beta \) for limit ordinals \( \alpha \). Moreover, for each nonlimit ordinal \( \alpha \), \( H_\alpha \) is a half-tranche of the irreducible continuum \([z_\alpha, w_\alpha]\).

PROOF. We will in fact define two sequences with this property, \([a_1, c_1] \supset A_1 \supset [a_2, c_2] \supset A_2 \supset \cdots \) and \([b_1, d_1] \supset B_1 \supset [b_2, d_2] \supset B_2 \supset \cdots \). We proceed by induction. Each \( A_\alpha \) and \( B_\alpha \) will be given an initial definition and will in some cases be redefined (at most once) during the induction. Under the initial definition \( A_\alpha \) and \( B_\alpha \) will always be (not necessarily distinct) half-tranches of \( T_\alpha \) with \( T_\alpha = A_\alpha \cup B_\alpha \). Let \([z_1, w_1] = [x_1, y_1]\) and let \( L \) and \( R \) be the two halves of \( T_1 \). If both are nondegenerate, let \( A_1 = L \) and \( B_1 = R \). If (without loss of generality) \( R \) is degenerate (i.e., \( R \) is empty or is a point), let \( A_1 = L = B_1 \). Note that in either case we have \( T_1 = A_1 \cup B_1 \).

Now suppose that \( A_\beta \) and \( B_\beta \) have been defined for all ordinals \( \beta \) less than some ordinal \( \alpha < \alpha_0 \). If \( \alpha = \beta + 1 \) for some ordinal \( \beta \), then \( T_\beta = A_\beta \cup B_\beta \) (no redefinition
THE DEPTH OF TRANCHES IN \( \lambda \)-DENDROIDS

has occurred yet). Let \( L_{\alpha} \) and \( R_{\alpha} \) be the two halves of \( T_{\alpha} \). We consider various cases.

Case 1. \( [x_{\alpha}, y_{\alpha}] \subset A_{\beta} \) (or similarly \( [x_{\alpha}, y_{\alpha}] \subset B_{\beta} \)). Then redefine all previous continua \( B_\gamma \) and \( [b_\gamma, d_\gamma] \) by setting \( B_\gamma = A_\gamma \) and \( [b_\gamma, d_\gamma] = [a_\gamma, c_\gamma] \) for all \( \gamma \leq \beta \). Note that since we now have \( A_\gamma = B_\gamma \) and \( [a_\gamma, c_\gamma] = [b_\gamma, d_\gamma] \) for all \( \gamma \leq \beta \), any further redefinitions of this type will not change these continua. This same remark applies to all of the redefinitions given below. If \( L_{\alpha} \) and \( R_{\alpha} \) are both nondegenerate, let \( A_{\alpha} = L_{\alpha} \) and \( B_{\alpha} = R_{\alpha} \). If (without loss of generality) \( R_{\alpha} \) is degenerate, let \( A_{\alpha} = L_{\alpha} = B_{\alpha} \). In either case, let \( [a_\alpha, c_\alpha] = [x_{\alpha}, y_{\alpha}] = [b_\alpha, d_\alpha] \).

Case 2. \( [x_{\alpha}, y_{\alpha}] \) is contained in neither \( A_{\beta} \) nor \( B_{\beta} \). Then \( x_{\alpha} \) and \( y_{\alpha} \) cannot both lie in \( A_\beta \) or \( B_\beta \). So suppose (without loss of generality) that \( x_{\alpha} \in A_\beta \) and \( y_{\alpha} \in B_\beta \). Let \( s \in A_\beta \cap B_\beta \cap [x_{\alpha}, y_{\alpha}] \). Then \( [x_{\alpha}, s] \cup [s, y_{\alpha}] = [x_{\alpha}, y_{\alpha}] \), \( [x_{\alpha}, s] \subset A_\beta \) and \( [s, y_{\alpha}] \subset B_\beta \). Let \( L_{\alpha} \) and \( R_{\alpha} \) be the two halves of \( T_{\alpha} \). By 2.2 each of these continua is a half-tranche of either \( [x_{\alpha}, s] \) or \( [s, y_{\alpha}] \).

Case 2a. \( L_{\alpha} \) and \( R_{\alpha} \) are both half-tranches of \( [x_{\alpha}, s] \) (or similarly, both are half-tranches of \( [s, y_{\alpha}] \)). If both \( L_{\alpha} \) and \( R_{\alpha} \) are nondegenerate, let \( A_{\alpha} = L_{\alpha} \) and \( B_{\alpha} = R_{\alpha} \). If (without loss of generality) \( R_{\alpha} \) is degenerate, let \( A_{\alpha} = L_{\alpha} = B_{\alpha} \). In either case let \( [a_{\alpha}, c_{\alpha}] = [x_{\alpha}, s] = [b_{\alpha}, d_{\alpha}] \) and redefine all previous \( B_\gamma \) and \( [b_\gamma, d_\gamma] \) by setting \( B_\gamma = A_\gamma \) and \( [b_\gamma, d_\gamma] = [a_\gamma, c_\gamma] \) for all \( \gamma \leq \beta \).

Case 2b. Case 2a fails. Then we may assume without loss of generality that \( L_{\alpha} \) is a half-tranche of \( [x_{\alpha}, s] \) and \( R_{\alpha} \) is a half-tranche of \( [s, y_{\alpha}] \). If both \( L_{\alpha} \) and \( R_{\alpha} \) are both nondegenerate, set \( A_{\alpha} = L_{\alpha} \), \( [a_\alpha, c_\alpha] = [x_\alpha, s] \), \( B_{\alpha} = R_{\alpha} \) and \( [b_\alpha, d_\alpha] = [s, y_\alpha] \). If (without loss of generality) \( R_{\alpha} \) is degenerate, set \( A_{\alpha} = L_{\alpha} = B_{\alpha} \) and redefine all previous continua \( B_\gamma \) and \( [b_\gamma, d_\gamma] \) by setting \( B_\gamma = A_\gamma \) and \( [b_\gamma, d_\gamma] = [a_\gamma, c_\gamma] \) for all \( \gamma \leq \beta \). These complete the definitions for \( \alpha \) a nonlimit ordinal.

Now suppose \( \alpha \) is a limit ordinal. We need only define \( A_{\alpha} \) and \( B_{\alpha} \). We consider two cases.

Case 1. For cofinally many \( \beta < \alpha \), \( T_\beta = A_\beta \cup B_\beta \). Let \( L_{\alpha} = \bigcap_{\beta < \alpha} A_\beta \) and \( R_{\alpha} = \bigcap_{\beta < \alpha} B_\beta \). Then, since the \( A_\beta \) and \( B_\beta \) are nested, we have \( T_{\alpha} = L_{\alpha} \cup R_{\alpha} \). If \( L_{\alpha} \) and \( R_{\alpha} \) are both nondegenerate, set \( A_{\alpha} = L_{\alpha} \) and \( B_{\alpha} = R_{\alpha} \). If (without loss of generality) \( R_{\alpha} \) is degenerate, set \( A_{\alpha} = L_{\alpha} = B_{\alpha} \) and redefine the previous \( B_\gamma \) and \( [b_\gamma, d_\gamma] \) as above. Note that we still have \( T_{\alpha} = A_{\alpha} \cup B_{\alpha} \).

Case 2. \( A_\beta = B_\beta \) for all \( \beta < \alpha \) and Case 1 fails. It follows that cofinally many redefinitions have occurred. (Recall, however, that any individual \( A_\beta \) and \( B_\beta \) will be changed at most once.) It is not difficult to verify that, however the redefinitions occurred, there will be cofinally many \( \beta < \alpha \) for which \( T_{\beta+1} \subset A_\beta \). It follows that \( \bigcap_{\beta < \alpha} A_\beta = \bigcap_{\beta < \alpha} B_\beta = T_\beta = T_{\alpha} \). So let \( A_{\alpha} = T_{\alpha} = B_{\alpha} \). It is not difficult to verify that Cases 1 and 2 exhaust the possibilities for limit ordinals \( \alpha \). Q.E.D.

We now proceed to the proof of the main theorem. The proof will make use of the constructions introduced by Krasinkiewicz and Minc in [6]. The presentation given here is slightly simplified for our particular setting.

**DEFINITION 2.4.** Let \( L \) be a \( \lambda \)-dendroid and let \( x, y \in L \). Let \( U \) and \( V \) be open subsets of \( L \). We say that the triple \( (L, x, y) \) is crooked between \( U \) and \( V \) and write \((x, y) \in \text{cr}(U, V)\) if there is a chain of closed sets \( \{F_1, F_2, F_3\} \) such that

\[
L = F_1 \cup F_2 \cup F_3, \quad x \in F_1 \cap U, \quad F_1 \cap F_2 \subset V, \\
F_2 \cap F_3 \subset U, \quad \text{and} \quad y \in F_3 \cap V.
\]
Given a $\lambda$-dendroid $L$ and a set of pairs $K \subset L \times L$, we define the set $K^c$ by $K^c = \{(x, y) \in L \times L: \text{given any neighborhoods } U \text{ of } x \text{ and } V \text{ of } y, \text{ there is a pair } (x', y') \in K \text{ with } x' \in U, y' \in V \text{ and } (x', y') \in \text{cr}(U, V)\}$.

**Definition 2.5.** Let $L$ be a $\lambda$-dendroid. Let $K = L \times L$. For each countable ordinal $\alpha$, we define a set $K_\alpha$ as follows: $K_1 = K^c$. If $\alpha = \beta + 1$, then $K_\alpha = (K_\beta)^c$. If $\alpha$ is a limit ordinal, then $K_\alpha = \bigcap_{\beta < \alpha} K_\beta$. The following lemma is a corollary of 2.7 in [6].

**Lemma 2.6.** For any $\lambda$-dendroid $L$, there is a countable ordinal $\alpha$ such that $K_\alpha = \emptyset$.

**Proposition 2.7.** Let $X$ be a $\lambda$-dendroid and suppose $X$ contains a transfinite sequence of half-tranches $[z_1, w_1] \supset H_1 \supset [z_2, w_2] \supset H_2 \supset \cdots$ as in 2.3. Then $K_\alpha \neq \emptyset$ for every $\alpha < \omega_0$.

**Proof.** We will show that $H_\alpha \times H_\alpha \subset K_\alpha$ for every $\alpha < \omega_0$. So first consider $H_1$. Let $f^* : [z_1, w_1] \to [0, 1]$ be as in 1.1. Assume, without loss of generality, that $H_1$ is a “left” half-tranche of $[z_1, w_1]$ so that $H_1 \subset \text{cl}(f^{-1}([0, t_0]))$ where $t_0 \in [0, 1]$ and $H_1 \subset f^{-1}((t_1))$. Let $x, y \in H_1$ and let $U$ and $V$ be neighborhoods of $x$ and $y$ respectively. Then there are points $t_i \in (0, t_0)$, $i = 1, 2, 3, 4$, such that $0 < t_1 < t_2 < t_3 < t_4 < t_5$, $U \cap f^{-1}(t_1) \neq \emptyset$, $U \cap f^{-1}(t_3) \neq \emptyset$, $V \cap f^{-1}(t_2) \neq \emptyset$. For $i = 1, 2, 3, 4$ choose $a_i, b_i \in (0, t_0)$ such that $t_i \in (a_i, b_i)$ and the closed intervals $[a_i, b_i]$ are disjoint from one another. Now consider the continua $T_i$, $i = 0, 1, \ldots, 5$ defined as follows: for $i = 1, \ldots, 4$, $T_i = f^{-1}((a_i, b_i))$, $T_0 = f^{-1}(0)$ and $T_5 = H_1$. Then for any $i, j, k$, $i < j < k$, with $i < j < k < l$, if $C$ is a subcontinuum of $[z_1, w_1]$ which meets $T_i$ and $T_k$, it must contain $T_j$. Moreover $[z_1, w_1]$ is tree-like (see [3]). It follows that there is a closed tree-chain cover $\mathcal{T}$ of $[z_1, w_1]$ whose union is $[z_1, w_1]$ (indeed any tree-chain cover of sufficiently small mesh will satisfy the following condition) and containing a chain $\mathcal{C} = \{U_0, U_1, \ldots, U_n\}$ such that $U_0 \cap T_0 \neq \emptyset$, $U_n \cap T_5 \neq \emptyset$ and $\mathcal{C}$ contains links $U_i, U_j, U_k, U_l$ with $i < j < k < l$ and $U_i \subset T_1 \cap T_0$, $U_j \subset T_4 \cap V$, $U_k \subset T_3 \cap U$ and $U_l \subset T_4 \cap V$. Let $F_1 = \bigcup\{U_0, \ldots, U_j\}$, $F_2 = \bigcup\{U_{j+1}, \ldots, U_k\}$ and $F_3 = \bigcup\{U_{k+1}, \ldots, U_n\}$. Let $x' \in U_i$ and $y' \in U_l$. It is not difficult to verify that $F_1, F_2, F_3$ satisfies the desired “crookedness pattern” with respect to $x', y', U$ and $V$. We will expand these three sets, first to a chain covering $[z_1, w_1]$ and then to a chain covering $X$.

Any link of $\mathcal{T}$ which is not in the chain $\mathcal{C}$ can be joined to $\mathcal{C}$ by a chain that meets exactly one of the sets $F_i$, $i = 1, 2, 3$. By adjoining the unions of these chains to the appropriate sets $F_i$, we obtain an expanded chain $F''_1, F''_2, F''_3$ whose union is $[z_1, w_1]$ and with the same crookedness pattern as $F_1, F_2, F_3$. Finally, apply Theorem 2.1 of [6] to obtain the desired chain $\{F_1, F_2, F_3\}$ whose union is $X$.

Now suppose that we have $H_\beta \times H_\beta \subset K_\beta$ for every $\beta$ less than some ordinal $\alpha$. Suppose first that $\alpha = \beta + 1$ for some $\beta$. Let $x, y \in H_\alpha$ and let $U$ and $V$ be neighborhoods of $x$ and $y$ respectively. By a proof similar to the above, we may find points $x', y' \in [z_\alpha, w_\alpha]$ and a chain $F''_1, F''_2, F''_3$ whose union is $[z_\alpha, w_\alpha]$ and satisfying the desired crookedness condition with respect to $x', y', U$ and $V$. Applying Theorem 2.1 of [6] again, we may expand the sets $F''_1, F''_2, F''_3$ to obtain the desired chain $\{F_1, F_2, F_3\}$ whose union is $X$. Since $x', y' \in [z_\alpha, w_\alpha] \subset H_\beta$, we have $(x, y) \in K_\beta$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Finally suppose $\alpha$ is a limit ordinal. Then $H_\alpha = \bigcap_{\beta < \alpha} H_\beta$. So we have directly from the definition that $H_\alpha \times H_\alpha \subset K_\alpha$. Q.E.D.

**Corollary 2.8.** Let $X$ be a $\lambda$-dendroid. Then there is a countable ordinal $\tau(X)$ which is an upper bound on the length of all sequences of tranches as in Lemma 2.3.

**Corollary 2.9.** Let $C$ be a hereditarily decomposable chainable continuum. Then there is a countable ordinal upper bound on the length of sequences $\{T_\alpha\}$ of nondegenerate subcontinua of $C$ such that (i) $T_1$ is a tranche of $C$, (ii) for each $\alpha = \beta + 1$, $T_\alpha$ is a tranche of $T_\beta$, and (iii) for limit ordinals $\alpha$, $T_\alpha = \bigcap_{\beta < \alpha} T_\beta$.

Corollary 2.9 answers the question of Thomas mentioned in the Introduction (see [8, p. 58]).

3. Normal sequences. In [5] Illiadis defines the notion of a normal sequence for hereditarily decomposable continua as follows. Let $X$ be a hereditarily decomposable continuum. A continuum $H \subset X$ is said to be in $I(X)$ if given any decomposition of $X$ into finitely many subcontinua, $H$ is contained in one element of the decomposition. Let $\{H_\alpha\}_{\alpha < \alpha_0}$ be a transfinite sequence of subcontinua of $X$, where $\alpha_0$ is some countable ordinal. $\{H_\alpha\}_{\alpha < \alpha_0}$ is called a normal sequence if (i) $H_0 = X$, (ii) for ordinals $\alpha = \beta + 1$, $H_\alpha \in I(H_\beta)$, (iii) for limit ordinals $\alpha$, $H_\alpha = \bigcap_{\beta < \alpha} H_\beta$, and (iv) for each $\alpha < \alpha_0$, $H_\alpha$ is nondegenerate. The least upper bound of the lengths of all normal sequences in $X$ is denoted $k(X)$. Let $\Omega$ denote the first uncountable ordinal. Let $X$ be an arbitrary hereditarily decomposable continuum. Illiadis has asked (1) Is $k(X) < \Omega$? and (2) If $X$ is chainable, is $k(X) < \Omega$? It is not difficult to show that sequences of half-tranches as in 2.3 (delete the continua $[z_0, w_\alpha]$) are normal sequences. Therefore, for $\lambda$-dendroids we have $\tau(X) \leq k(X)$. Below we show that for chainable continua this inequality can be reversed, thus answering Illiadis’s question for this particular class. Note that chainable continua are irreducible. Moreover, any subcontinuum of a chainable continuum is chainable and hence irreducible. Thus any subcontinuum of a hereditarily decomposable chainable continuum admits a decomposition into tranches.

**Proposition 3.1.** Let $C$ be a hereditarily decomposable chainable continuum and suppose $\{K_\alpha\}_{\alpha < \alpha_0}$ is a normal sequence of subcontinua of $C$. Then there is a sequence $\{T_\alpha\}_{\alpha < \alpha_0}$ of tranches-within-tranches of $C$ such that $K_\alpha \subset T_\alpha$ for every $\alpha$. Thus $\tau(C) \geq k(C)$.

**Proof.** Let $\{K_\alpha\}_{\alpha < \alpha_0}$ be given. Let $T_0 = C = K_0$. Now suppose that the continua $T_\beta$ have been defined for every $\beta$ less than some ordinal $\alpha < \alpha_0$ and $K_\beta \subset T_\beta$ for all such $\beta$. If $\alpha$ is a limit ordinal, then $T_\alpha = \bigcap_{\beta < \alpha} T_\beta \supset \bigcap_{\beta < \alpha} K_\beta = K_\alpha$. Suppose $\alpha = \beta + 1$ for some ordinal $\beta$. Then it is not difficult to see that $K_\alpha$ must lie in some tranche (indeed in a half-tranche) of $T_\beta$. For otherwise $T_\beta$ would admit a decomposition $A \cup B$ such that $K_\alpha$ would not be contained in either $A$ or $B$. But then $A \cap K_\beta$ and $B \cap K_\beta$ would produce a similar decomposition of $K_\beta$, a contradiction. So call this tranche of $T_\beta$ $T_\alpha$, and we are done. Q.E.D.

4. Remarks and questions. It is not difficult to give examples of $\lambda$-dendroids (in fact dendroids) $X$ such that $\tau(X) < k(X)$. Thus Illiadis’s question remains open for $\lambda$-dendroids as well as arbitrary hereditarily decomposable continua. Note that hereditary unicoherence is not required in defining $T$-sequences inside a $\lambda$-dendroid.
We may also carry out the construction inside any hereditarily decomposable continuum. (At each stage of the construction, let \([x_\alpha, y_\alpha]\) denote any continuum irreducible between \(x_\alpha\) and \(y_\alpha\).) Thus we may ask the following question.

**QUESTION 4.1.** Let \(X\) be a hereditarily decomposable continuum. Is \(\tau(X) < \Omega\) ?

Iliadis has also constructed in [5] a collection of chainable continua \(Q_\alpha\), one for each countable ordinal \(\alpha\), such that \(k(Q_\alpha) = \alpha\). In light of 3.1, we also have \(\tau(Q_\alpha) = \alpha\). Thus there exists a family of chainable continua containing tranches of any desired depth. For the benefit of readers who may not have easy access to [5] we describe here, without proof, a similar collection of chainable continua \(X_\alpha\).

**DEFINITION 4.2.** A mapping \(f: X \to Y\) of continua is said to be atomic if for any subcontinuum \(G\) of \(X\) such that \(f(G)\) is nondegenerate, we have \(f^{-1}(f(G)) = G\). Atomic maps were first defined by Cook [2], who called such mappings preatomic. Emeryk and Horbanowicz later [4], showed these maps are monotone and therefore "atomic" in Cook's sense.

**LEMMA 4.3.** Let \(X\) and \(Y\) be continua and let \(x_0 \in X\). Then there is a continuum \(Z\) which is a compactification of \(X - \{x_0\}\) and such that the complement of \(X - \{x_0\}\) in \(Z\) is homeomorphic to \(Y\). Moreover, the natural projection of \(Z\) onto \(X\) is atomic.

**EXAMPLE 4.4.** For each \(\alpha < \Omega\), let \([x_\alpha, y_\alpha]\) denote a homeomorphic copy of the closed interval \([0, 1]\). The interval \([x_\alpha, y_\alpha]\) will always be a subcontinuum of the space \(X_\alpha\), and the point \(x_\alpha\) will always be a point of irreducibility of \(X_\alpha\). For \(\beta < \alpha\), we will also have atomic maps \(f_\beta^\alpha: X_\alpha \to X_\beta\) such that \(f_\beta^\alpha(x_\alpha) = x_\beta\). Let \(X_1 = [x_1, y_1]\). We define the remaining spaces \(X_\alpha\) by induction. Suppose that the continua \(X_\beta\) have been defined for all \(\beta < \alpha\). If \(\alpha = \beta + 1\) for some \(\beta\), let \(X_\alpha\) be a compactification of \(X_\beta - \{x_\beta\}\) with remainder \([x_\alpha, y_\alpha]\) and such that the natural projection \(f_\beta^\alpha(X_\alpha) \to X_\beta\) is atomic. For limit ordinals \(\alpha\), let \(X_\alpha\) denote the inverse limit of the previously defined continua \(X_\beta\) with bonding maps the previously defined \(f_\beta^\alpha\). For \(\beta < \alpha\), let \(f_\beta^{\alpha'}\) denote the projection of \(X_\alpha\) onto the factor space \(X_\beta\). Let \(x_\alpha\) denote the point \((x_1, x_2, \ldots)\) in the inverse limit space \(X_\alpha\). Let \(X_\alpha\) be a compactification of \(X_\alpha' - \{x_\alpha\}\) with remainder \([x_\alpha, y_\alpha]\) and such that the natural projection \(f_\alpha^\alpha: X_\alpha \to X_\alpha'\) is atomic. For \(\beta < \alpha\), let \(f_\beta = f_\beta^{\alpha'} \circ f_\alpha^\alpha\).

**REFERENCES**

8. E. S. Thomas, Jr., *Monotone decompositions of irreducible continua*, Rozprawy Mat. **50** (1966), 74 pages.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA, BIRMINGHAM, ALABAMA 35294