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THE DEPTH OF TRANCHES IN λ-DENDROIDS

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ABSTRACT. According to the well-known theory of Kuratowski, any hereditarily decomposable chainable continuum admits a decomposition into tranches. These tranches are themselves chainable and thus admit decompositions into their own tranches. We may thus define nested sequences $\{T_\alpha\}$ of tranches-within-tranches, indexed by countable ordinals α, and finally terminating in a singleton set. E. S. Thomas, Jr. has asked whether, for a given continuum C, there is a countable ordinal bound on the length of all such nests $\{T_\alpha\}$ in C. We answer Thomas's question in the affirmative. By generalizing the definitions, we obtain the same result for λ-dendroids. We also answer, for chainable continua, a related question of Illiadis.

1. Introduction. According to the well-known theory of Kuratowski, any hereditarily decomposable chainable continuum admits a decomposition into tranches. (Technical terms will be defined below.) These tranches are themselves chainable continua and consequently admit decompositions into tranches. We may thus define nests $\{T_\alpha\}$ of tranches-within-tranches, indexed by countable ordinals α. Any individual such nest must (by metrizability) end in a point for some countable ordinal α. In [**8**] E. S. Thomas, Jr. raised the question whether, for a given chainable continuum C, there is a countable ordinal bound on the length of all nests of tranches-within-tranches in C. In this paper we answer Thomas's question in the affirmative (Corollary 2.9 below). Indeed, by modifying the definition slightly, we obtain the same result for the wider class of λ-dendroids (Corollary 2.8 below). The proof applies the "crookedness index" of Krasinkiewicz and Minc [**6**] to sequences of "half-tranches" obtained in Lemma 2.3 below, thus limiting their length. The author is indebted to Lex Oversteegen for suggesting the usefulness of [**6**] in attacking Thomas's problem and for observing the generalizability of the author's original proof from chainable continua to λ-dendroids. In §3 we apply the results of §2 to obtain an affirmative answer to a question of Illiadis [**5**] on nests of continua in hereditarily decomposable chainable continua. §4 is devoted to questions and examples.

A λ-*dendroid* is a hereditarily decomposable, hereditarily unicoherent (metric) continuum. Given a λ-dendroid X and points $x, y \in X$, there is a unique subcontinuum $[x, y]$ of X, irreducible with respect to containing both x and y. By Kuratowski's structure theory for irreducible continua (see [**7**, §48]; see also [**8**]), we have the following theorem.

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THEOREM 1.1. *Let X be a λ-dendroid and let $x, y \in X$ with $x \neq y$. Then there is a monotone mapping $f^*\colon [x,y] \to [0,1]$ such that if $f\colon [x,y] \to [0,1]$ is any other monotone map, then there is a monotone map $m\colon [0,1] \to [0,1]$ such that $f = m \circ f^*$. Moreover, for every $t \in [0,1]$, $f^{*-1}(t)$ has void interior in $[x,y]$.*

The continua $f^{*-1}(t)$ in the above theorem are called *tranches* of the continuum $[x,y]$. It is not difficult to verify that if $f'\colon [x,y] \to [0,1]$ is any other monotone map with every set $f'^{-1}(t)$ having void interior in $[x,y]$, then the map m of 1.1 is a homeomorphism. Thus the decomposition of $[x,y]$ into tranches is unique.

Let X be a λ-dendroid. We may define a nested transfinite sequence of subcontinua of X as follows: Let $T_0 = X$. For successor ordinals $\alpha = \beta + 1$, if T_β is nondegenerate, let x_α and y_α be distinct points of T_β and let T_α be a tranche of $[x_\alpha, y_\alpha]$. For limit ordinals α, let $T_\alpha = \bigcap_{\beta<\alpha} T_\beta$. Note that we also have $T_\alpha = \bigcap_{\beta<\alpha}[x_\beta, y_\beta]$. As soon as some T_α is a degenerate continuum, we stop the sequence. Since X is metrizable, any given sequence like the above must stop at some countable ordinal α. We will show below that in fact there is a countable ordinal $\tau(X)$ such that every sequence of tranches-within-tranches defined as above in X, stops at or before $\tau(X)$. In what follows we will assume some familiarity with Kuratowski's structure theory for irreducible continua. (See [**7**, §48] or [**8**].)

2. Sequences of tranches and half-tranches.

DEFINITION 2.1. Let X be a λ-dendroid, $x, y \in X$ with $x \neq y$ and let $f^*\colon [x,y] \to [0,1]$ be as in Theorem 1.1. Let $t_0 \in [0,1]$ and $T = f^{*-1}(t_0)$ be the associated tranche of $[x,y]$. Let $L = T \cap \mathrm{cl}(f^{*-1}([0,t_0)))$ and $R = T \cap \mathrm{cl}(f^{*-1}((t_0,1]))$. (If $t_0 = 0$ or 1, one of these sets will be empty.) Then $T = L \cup R$. L and R will be called *halves* of T and *half-tranches* of $[x,y]$.

The proof of the following lemma is straightforward.

LEMMA 2.2. *Let X be a λ-dendroid, $x, y \in X$ with $x \neq y$ and let L and R be the two halves of the tranche T of $[x,y]$. Then if z is any point of $[x,y]$, L is a half-tranche of either $[x,z]$ or $[z,y]$. Similarly for R.*

LEMMA 2.3. *Let $[x_1,y_1] \supset T_1 \supset [x_2,y_2] \supset \cdots$ be a transfinite sequence defined as in the last paragraph of* §1. *Suppose that for some countable ordinal α_0, we have T_α nondegenerate for each $\alpha < \alpha_0$. Then there is a sequence $[z_1,w_1] \supset H_1 \supset [z_2,w_2] \supset H_2 \supset \cdots$ with H_α nondegenerate for each $\alpha < \alpha_0$ and $H_\alpha = \bigcap_{\beta<\alpha} H_\beta$ for limit ordinals α. Moreover, for each nonlimit ordinal α, H_α is a half-tranche of the irreducible continuum $[z_\alpha, w_\alpha]$.*

PROOF. We will in fact define two sequences with this property, $[a_1,c_1] \supset A_1 \supset [a_2,c_2] \supset A_2 \supset \cdots$ and $[b_1,d_1] \supset B_1 \supset [b_2,d_2] \supset B_2 \supset \cdots$. We proceed by induction. Each A_α and B_α will be given an initial definition and will in some cases be redefined (at most once) during the induction. Under the initial definition A_α and B_α will always be (not necessarily distinct) half-tranches of T_α with $T_\alpha = A_\alpha \cup B_\alpha$. Let $[z_1,w_1] = [x_1,y_1]$ and let L and R be the two halves of T_1. If both are nondegenerate, let $A_1 = L$ and $B_1 = R$. If (without loss of generality) R is degenerate (i.e., R is empty or is a point), let $A_1 = L = B_1$. Note that in either case we have $T_1 = A_1 \cup B_1$.

Now suppose that A_β and B_β have been defined for all ordinals β less than some ordinal $\alpha < \alpha_0$. If $\alpha = \beta + 1$ for some ordinal β, then $T_\beta = A_\beta \cup B_\beta$ (no redefinition has occurred yet). Let L_α and R_α be the two halves of T_α. We consider various cases.

Case 1. $[x_\alpha, y_\alpha] \subset A_\beta$ *(or similarly* $[x_\alpha, y_\alpha] \subset B_\beta$*).* Then redefine all previous continua B_γ and $[b_\gamma, d_\gamma]$ by setting $B_\gamma = A_\gamma$ and $[b_\gamma, d_\gamma] = [a_\gamma, c_\gamma]$ for all $\gamma \leq \beta$. Note that since we now have $A_\gamma = B_\gamma$ and $[a_\gamma, c_\gamma] = [b_\gamma, d_\gamma]$ for all $\gamma \leq \beta$, any further redefinitions of this type will not change these continua. This same remark applies to all of the redefinitions given below. If L_α and R_α are both nondegenerate, let $A_\alpha = L_\alpha$ and $B_\alpha = R_\alpha$. If (without loss of generality) R_α is degenerate, let $A_\alpha = L_\alpha = B_\alpha$. In either case, let $[a_\alpha, c_\alpha] = [x_\alpha, y_\alpha] = [b_\alpha, d_\alpha]$.

Case 2. $[x_\alpha, y_\alpha]$ *is contained in neither* A_β *nor* B_β. Then x_α and y_α cannot both lie in A_β or B_β. So suppose (without loss of generality) that $x_\alpha \in A_\beta$ and $y_\alpha \in B_\beta$. Let $s \in A_\beta \cap B_\beta \cap [x_\alpha, y_\alpha]$. Then $[x_\alpha, s] \cup [s, y_\alpha] = [x_\alpha, y_\alpha]$, $[x_\alpha, s] \subset A_\beta$ and $[s, y_\alpha] \subset B_\beta$. Let L_α and R_α be the two halves of T_α. By 2.2 each of these continua is a half-tranche of either $[x_\alpha, s]$ or $[s, y_\alpha]$.

Case 2a. L_α *and* R_α *are both half-tranches of* $[x_\alpha, s]$ *(or similarly, both are half-tranches of* $[s, y_\alpha]$*).* If both L_α and R_α are nondegenerate, let $A_\alpha = L_\alpha$ and $B_\alpha = R_\alpha$. If (without loss of generality) R_α is degenerate, let $A_\alpha = L_\alpha = B_\alpha$. In either case let $[a_\alpha, c_\alpha] = [x_\alpha, s] = [b_\alpha, d_\alpha]$ and redefine all previous B_γ and $[b_\gamma, d_\gamma]$ by setting $B_\gamma = A_\gamma$ and $[b_\gamma, d_\gamma] = [a_\gamma, c_\gamma]$ for all $\gamma \leq \beta$.

Case 2b. *Case* 2a *fails.* Then we may assume without loss of generality that L_α is a half-tranche of $[x_\alpha, s]$ and R_α is a half-tranche of $[s, y_\alpha]$. If L_α and R_α are both nondegenerate, set $A_\alpha = L_\alpha$, $[a_\alpha, c_\alpha] = [x_\alpha, s]$, $B_\alpha = R_\alpha$ and $[b_\alpha, d_\alpha] = [s, y_\alpha]$. If (without loss of generality) R_α is degenerate, set $A_\alpha = L_\alpha = B_\alpha$ and redefine all previous continua B_γ and $[b_\gamma, d_\gamma]$ by setting $B_\gamma = A_\gamma$ and $[b_\gamma, d_\gamma] = [a_\gamma, c_\gamma]$ for all $\gamma \leq \beta$. These complete the definitions for α a nonlimit ordinal.

Now suppose α is a limit ordinal. We need only define A_α and B_α. We consider two cases.

Case 1. *For cofinally many* $\beta < \alpha$, $T_\beta = A_\beta \cup B_\beta$. Let $L_\alpha = \bigcap_{\beta<\alpha} A_\beta$ and $R_\alpha = \bigcap_{\beta<\alpha} B_\beta$. Then, since the A_β and B_β are nested, we have $T_\alpha = L_\alpha \cup R_\alpha$. If L_α and R_α are both nondegenerate, set $A_\alpha = L_\alpha$ and $B_\alpha = R_\alpha$. If (without loss of generality) R_α is degenerate, set $A_\alpha = L_\alpha = B_\alpha$ and redefine the previous B_β and $[b_\beta, d_\beta]$ as above. Note that we still have $T_\alpha = A_\alpha \cup B_\alpha$.

Case 2. $A_\beta = B_\beta$ *for all* $\beta < \alpha$ *and Case* 1 *fails.* It follows that cofinally many redefinitions have occurred. (Recall, however, that any individual A_β and B_β will be changed at most once.) It is not difficult to verify that, however the redefinitions occurred, there will be cofinally many $\beta < \alpha$ for which $T_{\beta+1} \subset A_\beta$. It follows that $\bigcap_{\beta<\alpha} B_\beta = \bigcap_{\beta<\alpha} A_\beta = \bigcap_{\beta<\alpha} T_\beta = T_\alpha$. So let $A_\alpha = T_\alpha = B_\alpha$. It is not difficult to verify that Cases 1 and 2 exhaust the possibilities for limit ordinals α. Q.E.D.

We now proceed to the proof of the main theorem. The proof will make use of the constructions introduced by Krasinkiewicz and Minc in [**6**]. The presentation given here is slightly simplified for our particular setting.

DEFINITION 2.4. Let L be a λ-dendroid and let $x, y \in L$. Let U and V be open subsets of L. We say that the triple (L, x, y) is *crooked between* U *and* V and write $(x, y) \in \mathrm{cr}(U, V)$ if there is a chain of closed sets $\{F_1, F_2, F_3\}$ such that

$$L = F_1 \cup F_2 \cup F_3, \qquad x \in F_1 \cap U, \qquad F_1 \cap F_2 \subset V,$$
$$F_2 \cap F_3 \subset U, \quad \text{and} \quad y \in F_3 \cap V.$$

Given a λ-dendroid L and a set of pairs $K \subset L \times L$, we define the set K^c by $K^c = \{(x, y) \in L \times L$: given any neighborhoods U of x and V of y, there is a pair $(x', y') \in K$ with $x' \in U$, $y' \in V$ and $(x', y') \in \mathrm{cr}(U, V)\}$.

DEFINITION 2.5. Let L be a λ-dendroid. Let $K = L \times L$. For each countable ordinal α, we define a set K_α as follows: $K_1 = K^c$. If $\alpha = \beta + 1$, then $K_\alpha = (K_\beta)^c$. If α is a limit ordinal, then $K_\alpha = \bigcap_{\beta<\alpha} K_\beta$. The following lemma is a corollary of 2.7 in [**6**].

LEMMA 2.6. *For any λ-dendroid L, there is a countable ordinal α such that $K_\alpha = \varnothing$.*

PROPOSITION 2.7. *Let X be a λ-dendroid and suppose X contains a transfinite sequence of half-tranches $[z_1, w_1] \supset H_1 \supset [z_2, w_2] \supset H_2 \supset \cdots$ as in 2.3. Then $K_\alpha \neq \varnothing$ for every $\alpha < \alpha_0$.*

PROOF. We will show that $H_\alpha \times H_\alpha \subset K_\alpha$ for every $\alpha < \alpha_0$. So first consider H_1. Let $f^*\colon [z_1, w_1] \to [0,1]$ be as in 1.1. Assume, without loss of generality, that H_1 is a "left" half-tranche of $[z_1, w_1]$ so that $H_1 \subset \mathrm{cl}(f^{*-1}([0, t_5)))$ where $t_5 \in [0,1]$ and $H_1 \subset f^{*-1}(t_5)$. Let $x, y \in H_1$ and let U and V be neighborhoods of x and y respectively. Then there are points $t_i \in (0, t_5)$, $i = 1,2,3,4$, such that $0 < t_1 < t_2 < t_3 < t_4 < t_5$, $U \cap f^{*-1}(t_1) \neq \varnothing \neq U \cap f^{*-1}(t_3)$ and $V \cap f^{*-1}(t_2) \neq \varnothing \neq f^{*-1}(t_4) \cap V$. For $i = 1,2,3,4$ choose $a_i, b_i \in (0, t_5)$ such that $t_i \in (a_i, b_i)$ and the closed intervals $[a_i, b_i]$ are disjoint from one another. Now consider the continua T_i, $i = 0, 1, \ldots, 5$ defined as follows: for $i = 1, \ldots, 4$, $T_i = f^{*-1}([a_i, b_i])$, $T_0 = f^{*-1}(0)$ and $T_5 = H_1$. Then for any $i, j, k = 0, 1, \ldots, 5$ with $i < j < k$, if C is a subcontinuum of $[z_1, w_1]$ which meets T_i and T_k, it must contain T_j. Moreover $[z_1, w_1]$ is tree-like (see [**3**]). It follows that there is a closed tree-chain cover $\mathcal{T}$ of $[z_1, w_1]$ whose union is $[z_1, w_1]$ (indeed any tree-chain cover of sufficiently small mesh will satisfy the following condition) and containing a chain $\mathcal{C} = \{U_0, U_1, \ldots, U_n\}$ such that $U_0 \cap T_0 \neq \varnothing$, $U_n \cap T_5 \neq \varnothing$ and $\mathcal{C}$ contains links U_i, U_j, U_k, U_l with $i < j < k < l$ and $U_i \subset T_1 \cap U$, $U_j \subset T_2 \cap V$, $U_k \subset T_3 \cap U$ and $U_l \subset T_4 \cap V$. Let $F_1' = \bigcup\{U_0, \ldots, U_j\}$, $F_2' = \bigcup\{U_{j+1}, \ldots, U_k\}$ and $F_3' = \bigcup\{U_{k+1}, \ldots, U_n\}$. Let $x' \in U_i$ and $y' \in U_l$. It is not difficult to verify that F_1', F_2', F_3' satisfies the desired "crookedness pattern" with respect to x', y', U and V. We will expand these three sets, first to a chain covering $[z_1, w_1]$ and then to a chain covering X.

Any link of $\mathcal{T}$ which is not in the chain $\mathcal{C}$ can be joined to $\mathcal{C}$ by a chain that meets exactly one of the sets F_i', $i = 1,2,3$. By adjoining the unions of these chains to the appropriate sets F_1', we obtain an expanded chain F_1'', F_2'', F_3'' whose union is $[z_1, w_1]$ and with the same crookedness pattern as F_1', F_2', F_3'. Finally, apply Theorem 2.1 of [**6**] to obtain the desired chain $\{F_1, F_2, F_3\}$ whose union is X.

Now suppose that we have $H_\beta \times H_\beta \subset K_\beta$ for every β less than some ordinal α. Suppose first that $\alpha = \beta + 1$ for some β. Let $x, y \in H_\alpha$ and let U and V be neighborhoods of x and y respectively. By a proof similar to the above, we may find points $x', y' \in [z_\alpha, w_\alpha]$ and a chain F_1'', F_2'', F_3'' whose union is $[z_\alpha, w_\alpha]$ and satisfying the desired crookedness condition with respect to x', y', U and V. Applying Theorem 2.1 of [**6**] again, we may expand the sets F_1'', F_2'', F_3'' to obtain the desired chain $\{F_1, F_2, F_3\}$ whose union is X. Since $x', y' \in [z_\alpha, w_\alpha] \subset H_\beta$, we have $(x, y) \in K_\beta^c$.

Finally suppose α is a limit ordinal. Then $H_\alpha = \bigcap_{\beta<\alpha} H_\beta$. So we have directly from the definition that $H_\alpha \times H_\alpha \subset K_\alpha$. Q.E.D.

COROLLARY 2.8. *Let X be a λ-dendroid. Then there is a countable ordinal $\tau(X)$ which is an upper bound on the length of all sequences of tranches as in Lemma* 2.3.

COROLLARY 2.9. *Let C be a hereditarily decomposable chainable continuum. Then there is a countable ordinal upper bound on the length of sequences $\{T_\alpha\}$ of nondegenerate subcontinua of C such that* (i) *T_1 is a tranche of C,* (ii) *for each $\alpha = \beta + 1$, T_α is a tranche of T_β, and* (iii) *for limit ordinals α, $T_\alpha = \bigcap_{\beta<\alpha} T_\beta$.*

Corollary 2.9 answers the question of Thomas mentioned in the Introduction (see [**8**, p. 58]).

3. Normal sequences. In [**5**] Illiadis defines the notion of a *normal sequence* for hereditarily decomposable continua as follows. Let X be a hereditarily decomposable continuum. A continuum $H \subset X$ is said to be in $I(X)$ if given any decomposition of X into finitely many subcontinua, H is contained in one element of the decomposition. Let $\{H_\alpha\}_{\alpha<\alpha_0}$ be a transfinite sequence of subcontinua of X, where α_0 is some countable ordinal. $\{H_\alpha\}_{\alpha<\alpha_0}$ is called a *normal sequence* if (i) $H_0 = X$, (ii) for ordinals $\alpha = \beta + 1$, $H_\alpha \in I(H_\beta)$, (iii) for limit ordinals α, $H_\alpha = \bigcap_{\beta<\alpha} H_\beta$, and (iv) for each $\alpha < \alpha_0$, H_α is nondegenerate. The least upper bound of the lengths of all normal sequences in X is denoted $k(X)$. Let Ω denote the first uncountable ordinal. Let X be an arbitrary hereditarily decomposable continuum. Illiadis has asked (1) Is $k(X) < \Omega$? and (2) If X is chainable, is $k(X) < \Omega$? It is not difficult to show that sequences of half-tranches as in 2.3 (delete the continua $[z_\alpha, w_\alpha]$) are normal sequences. Therefore, for λ-dendroids we have $\tau(X) \leq k(X)$. Below we show that for chainable continua this inequality can be reversed, thus answering Illiadis's question for this particular class. Note that chainable continua are irreducible. Moreover, any subcontinuum of a chainable continuum is chainable and hence irreducible. Thus any subcontinuum of a hereditarily decomposable chainable continuum admits a decomposition into tranches.

PROPOSITION 3.1. *Let C be a hereditarily decomposable chainable continuum and suppose $\{K_\alpha\}_{\alpha<\alpha_0}$ is a normal sequence of subcontinua of C. Then there is a sequence $\{T_\alpha\}_{\alpha<\alpha_0}$ of tranches-within-tranches of C such that $K_\alpha \subset T_\alpha$ for every α. Thus $\tau(C) \geq k(C)$.*

PROOF. Let $\{K_\alpha\}_{\alpha<\alpha_0}$ be given. Let $T_0 = C = K_0$. Now suppose that the continua T_β have been defined for every β less than some ordinal $\alpha < \alpha_0$ and $K_\beta \subset T_\beta$ for all such β. If α is a limit ordinal, then $T_\alpha = \bigcap_{\beta<\alpha} T_\beta \supset \bigcap_{\beta<\alpha} K_\beta = K_\alpha$. Suppose $\alpha = \beta + 1$ for some ordinal β. Then it is not difficult to see that K_α must lie in some tranche (indeed in a half-tranche) of T_β. For otherwise T_β would admit a decomposition $A \cup B$ such that K_α would not be contained in either A or B. But then $A \cap K_\beta$ and $B \cap K_\beta$ would produce a similar decomposition of K_β, a contradiction. So call this tranche of T_β T_α, and we are done. Q.E.D.

4. Remarks and questions. It is not difficult to give examples of λ-dendroids (in fact dendroids) X such that $\tau(X) < k(X)$. Thus Illiadis's question remains open for λ-dendroids as well as arbitrary hereditarily decomposable continua. Note that hereditary unicoherence is not required in defining T-sequences inside a λ-dendroid.

We may also carry out the construction inside any hereditarily decomposable continuum. (At each stage of the construction, let $[x_\alpha, y_\alpha]$ denote any continuum irreducible between x_α and y_α.) Thus we may ask the following question.

QUESTION 4.1. Let X be a hereditarily decomposable continuum. Is $\tau(X) < \Omega$?

Illiadis has also constructed in [**5**] a collection of chainable continua Q_α, one for each countable ordinal α, such that $k(Q_\alpha) = \alpha$. In light of 3.1, we also have $\tau(Q_\alpha) = \alpha$. Thus there exists a family of chainable continua containing tranches of any desired depth. For the benefit of readers who may not have easy access to [**5**] we describe here, without proof, a similar collection of chainable continua X_α.

DEFINITION 4.2. A mapping $f\colon X \to Y$ of continua is said to be *atomic* if for any subcontinuum C of X such that $f(C)$ is nondegenerate, we have $f^{-1}(f(C)) = C$. Atomic maps were first defined by Cook [**2**], who called such mappings preatomic. Emeryk and Horbanowicz later [**4**], showed these maps are monotone and therefore "atomic" in Cook's sense.

LEMMA 4.3. *Let X and Y be continua and let $x_0 \in X$. Then there is a continuum Z which is a compactification of $X - \{x_0\}$ and such that the complement of $X - \{x_0\}$ in Z is homeomorphic to Y. Moreover, the natural projection of Z onto X is atomic.*

EXAMPLE 4.4. For each $\alpha < \Omega$, let $[x_\alpha, y_\alpha]$ denote a homeomorphic copy of the closed interval $[0,1]$. The interval $[x_\alpha, y_\alpha]$ will always be a subcontinuum of the space X_α, and the point x_α will always be a point of irreducibility of X_α. For $\beta < \alpha$, we will also have atomic maps $f^\alpha_\beta\colon X_\alpha \to X_\beta$ such that $f^\alpha_\beta(x_\alpha) = x_\beta$. Let $X_1 = [x_1, y_1]$. We define the remaining spaces X_α by induction. Suppose that the continua X_β have been defined for all $\beta < \alpha$. If $\alpha = \beta + 1$ for some β, let X_α be a compactification of $X_\beta - \{x_\beta\}$ with remainder $[x_\alpha, y_\alpha]$ and such that the natural projection $f^\alpha_\beta(X_\alpha) \to X_\beta$ is atomic. For limit ordinals α, let X'_α denote the inverse limit of the previously defined continua X_β with bonding maps the previously defined f^α_β. For $\beta < \alpha$, let $f^{\alpha'}_\beta$ denote the projection of X'_α onto the factor space X_β. Let x'_α denote the point $(x_1, x_2, \ldots)$ in the inverse limit space X'_α. Let X_α be a compactification of $X'_\alpha - \{x'_\alpha\}$ with remainder $[x_\alpha, y_\alpha]$ and such that the natural projection $f^\alpha\colon X_\alpha \to X'_\alpha$ is atomic. For $\beta < \alpha$, let $f^\alpha_\beta = f^{\alpha'}_\beta \circ f^\alpha$.

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