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# FINITELY PRESENTED MODULES OVER SEMIPERFECT RINGS

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ABSTRACT. Results of Bjork and Sabbagh are extended and generalized to provide a Krull-Schmidt theory over a general class of semiperfect rings which includes left perfect rings, right perfect rings, and semiperfect PI-rings whose Jacobson radicals are nil.

The object of this paper is to lay elementary foundations to the study of f.g. (i.e. finitely generated) modules over rings which are almost Artinian, with the main goal being a theory following the lines of the Azumaya-Krull-Remak-Schmidt-Wedderburn theorem (commonly called Krull-Schmidt); in other words we wish to show that a given f.g. module is a finite direct sum of indecomposable submodules whose endomorphism rings are local. Previous efforts in this direction include [**2**, **3**, **4**, **6**], and in particular the results here extend some results of [**2**, **4**, **6**]. The focus here will be on a "Fitting's lemma" approach applied to semiperfect rings, cf. Theorem 8.

We recall the definition from [**1**], which will be used as a standard reference. $R$ is *semiperfect* if its Jacobson radical $J$ is idempotent-lifting and $R/J$ is semisimple Artinian; equivalently every f.g. module $M$ has a *projective cover* (an epic map $\pi\colon P \to M$, where $P$ is projective and $\ker \pi$ is a small submodule of $P$). Projective covers are unique up to isomorphism by [**1**, Lemma 17.17]. In what follows, *module* means "left module".

PROPOSITION 1. *If $R$ is semiperfect, then every f.g. module $M$ is a finite direct sum of indecomposable submodules.*

PROOF. Let $\pi\colon P \to M$ be a projective cover. Then $P$ has an indecomposable decomposition of some length (cf. [**1**, Theorem 27.12]) and we show by induction on $t$ that $M$ also has an indecomposable decomposition of length $\leq t$. Indeed this is tautological if $M$ is indecomposable, so assume $M = M_1 \oplus M_2$. By [**1**, Lemma 17.17] there are projective covers $\pi_i\colon P_i \to M_i$, where $P_i$ are direct summands of $P$, and in fact $P_1 \oplus P_2 \approx P$ by [**1**, Exercise 15.1], so we can proceed inductively on $M_1$ and $M_2$. Q.E.D.

REMARK 2. By [**1**, Theorem 27.6] an f.g. $R$-module $M$ is a direct sum of (indecomposable) modules having local endomorphism iff $\mathrm{End}_R M$ is semiperfect, so we ask: For which modules $M$ is $\mathrm{End}_R M$ semiperfect? (This is why it is natural to study semiperfect rings $R$.) In [**4**, Example 2.1] Bjork found an example of a cyclic module $M = R/L$ over a semiprimary ring $R$ such that $E = \mathrm{End}_R M$ is not

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semiperfect (because $E/\mathrm{Jac}(E)$ is a commutative ring which is not a field). Note his example also yields an onto map $M \to M$ which is not one-to-one. Our main results will therefore be about *finitely presented* modules $M$, i.e., $M = F/K$ where $F$ is an f.g. free module and $K$ is f.g., but we shall also say what we can for $M$ merely f.g.

PROPOSITION 3. *Suppose $R$ is semiperfect and $J = \mathrm{Jac}(R)$ is nil. If $Ra = Ra^2$ for some $a$ in $R$, then $Ra$ is a direct summand of $R$, i.e., $Ra = Re$ for some idempotent $e$.*

PROOF. Write $a = ba$ for some $b$ in $Ra$. Then $ba = b^2a$, so $b^2 - b \in Ra \cap \mathrm{Ann}\, a$. Letting $\bar{\ }$ be the image in $R/J$ we have $\overline{Ra^2} = \overline{Ra}$. Right multiplication by $\bar{a}$ gives a surjection $\psi\colon \overline{Ra} \to \overline{Ra}$ which is an isomorphism since $\overline{R}$ is semisimple Artinian. Then $\overline{b^2 - b} \in \overline{Ra} \cap \mathrm{Ann}\, \bar{a} = \ker \psi = 0$, so $\bar{b}$ is idempotent in $\overline{R}$; hence $b^2 - b$ is nilpotent. For some $k$ we have $0 = (b - b^2)^k = b^k - p(b)b^{k+1}$ where $p(\lambda)$ is a polynomial in $\mathbf{Z}[\lambda]$, the sum of whose coefficients is 1.

Let $e = (p(b)b)^k$ be an idempotent in $R$ whose image in $\overline{R}$ is $\bar{b}$. Then $ea = a$ since $b^i a = a$ for all $i$, and clearly $e \in Ra$. Write $e = ra$. Of course, as $R$-modules, length $\overline{Re} \geq$ length $\overline{Rea}$ = length $\overline{Ra}$; since $\overline{Re} \leq \overline{Ra}$, we have $\overline{Re} = \overline{Ra}$. Also length $\overline{Rr} \geq$ length $\overline{Re}$. On the other hand,

$$(ere)a = er(ea) = era = e^2 = e,$$

so replacing $r$ by $ere$ we have $r \in eRe$. In particular $\overline{Rr} \leq \overline{Re}$ so $\overline{Rr} = \overline{Re}$. Write $\bar{e} = \overline{r'r} = \overline{r'er} = \overline{(er'e)(ere)}$. Thus $\bar{r}$ is invertible in $\overline{eRe}$ implying $\overline{r + 1 - e}$ is invertible in $\overline{R}$. Thus $r + 1 - e$ is invertible in $R$ (since invertibility lifts up the Jacobson radical). But $(r + 1 - e)a = ra + a - ea = ra = e$, implying $Ra = Re$, as desired. Q.E.D.

If we are to have a theory of modules satisfying a version of Fitting's lemma, then certainly $\mathrm{Jac}(R)$ must be nil when $R$ is local (taking $M = R$); in fact a somewhat stronger condition is needed, used in [**2**, **5**]:

A ring $R$ is called *left $\pi$-regular* if it satisfies the DCC on chains of the form $Ra > Ra^2 > Ra^3 > \cdots$. This condition is left-right symmetric by [**5**]. Note that if $R$ is left $\pi$-regular, then $\mathrm{Jac}(R)$ is nil (for if $a^n \in Ra^{n+1}$, then $a^n = ra^{n+1}$ for some $r$, implying $(1 - ra)a^n = 0$, and thus $a^n = 0$). On the other hand, [**2**, Proposition 2.3] shows that Fitting's lemma holds for $R$ (as an $R$-module) iff $R$ is left $\pi$-regular, so this class of rings is clearly of interest to us. Actually we are interested in a slightly stronger condition, in order to deal with arbitrary f.g. modules. Let us say $R$ is *$\pi_\infty$-regular* if $M_n(R)$ is left $\pi$-regular for each $n$. (Since this condition also is left-right symmetric we have dropped the word "left"; Dischinger [**5**] uses the terminology "completely $\pi$-regular".)

Any right perfect ring $R$ is $\pi_\infty$-regular since each matrix ring over $R$ is also right perfect and thus satisfies the descending chain condition on principal left ideals. Thus [**5**] implies any left perfect ring $R$ also is $\pi_\infty$-regular. There is an example of a semiperfect ring whose Jacobson radical is nil which is *not* $\pi_\infty$-regular, cf. [**10**], but such an example is rather hard to come by; a more thorough discussion is given in the appendix, which provides some positive results concerning when a given semiperfect ring is left $\pi$-regular.

REMARK 4. Suppose $f\colon M \to M$ and $g\colon M \to N$ are maps of $R$-modules with $gfM = gM$. Then $M = fM + \ker g$. (This is standard: If $x \in M$, then $gx = gfy$ for some $y$, so $x - fy \in \ker g$.)

REMARK 5. If $f\colon P \to M$ is a projective cover and $P = P_1 \oplus P_2$, then letting $f_i$ denote the restriction of $f$ to $P_i$ we have projective covers $f_i\colon P_i \to fP_i$ for $i = 1, 2$. (Indeed the $P_i$ are projective so it suffices to show $\ker f_i$ is small in $P_i$. If $N + \ker f_1 = P_1$, then $N + P_2 + \ker f = P$, implying $N + P_2 = P$, so $N = P_1$.)

PROPOSITION 6. *Suppose $R$ is a semiperfect ring with $M_n(\mathrm{Jac}(R))$ nil for all $n$ (e.g., this holds when $\mathrm{Jac}(R)$ is locally nilpotent). If $P$ is an f.g. projective $R$-module and $f\colon P \to P$ satisfies $fP = f^2P$, then $P = fP \oplus \ker f$. (In particular $f$ restricts to an isomorphism from $fP$ to itself.)*

PROOF. $P = fP + \ker f$ by Remark 4, so it suffices to prove $fP \cap \ker f = 0$. Writing $P \oplus P' = F$ f.g. free and extending $f$ to $F$ by putting $fx = x$ for all $x$ in $P'$, we may thereby assume $P$ is free. But $f$ now acts as right multiplication by some matrix $a$ in $M_n(R)$, viewing the elements of $P$ as row vectors. Filling in with zeros underneath to view $P \subset M_n(R)$ we apply Proposition 3 to $M_n(R)$ to see $M_n(R)a$ is a direct summand of $M_n(R)$ as an $M_n(R)$-module. Hence $Pa$ is a direct summand of $P$ as an $R$-module, implying $fP$ is a projective module, and $f\colon fP \to fP$ is onto and thus an isomorphism. (This is well known and has the following easy argument. As above we may assume $fP$ is free, so $f$ acts by right multiplication. This assertion is true for semisimple Artinian rings and thus for semilocal rings.) Thus $fP \cap \ker f = 0$, as desired. Q.E.D.

PROPOSITION 7. *Suppose $R$ is a semiperfect, $\pi_\infty$-regular ring. If $M$ is an f.g. $R$-module and $f\colon M \to M$ is a map, then $f^tM = f^{t+1}M$ for some $t$.*

PROOF. A standard trick enables us to assume $M$ is cyclic (for if $M$ is spanned by $n$ elements, i.e., $M = \sum_{i=1}^n Rx_i$, then $M^{(n)}$ is a cyclic $M_n(R)$-module generated by $(x_1, \dots, x_n)$, so we can replace $R$ by $M_n(R)$ and $M$ by $M^{(n)}$). But now writing $M = Rx$ we have $fx = ax$ for some $a$ in $R$ and $a^t \in Ra^{t+1}$ for some $t$ (by hypothesis), implying $f^tM \subseteq f^{t+1}M$, so $f^tM = f^{t+1}M$. Q.E.D.

THEOREM 8. *Suppose $R$ is a semiperfect, $\pi_\infty$-regular ring, and $M$ is a finitely presented $R$-module.*

(i) *If $f\colon M \to M$ satisfies $fM = f^2M$, then $M = fM \oplus \ker f$.*

(ii) "*Fitting's lemma*" : *If $M$ is indecomposable, then every endomorphism of $M$ is either invertible or nilpotent, so in particular $\mathrm{End}\, M$ is a local ring whose Jacobson radical is nil.*

(iii) *There is a decomposition $M = \bigoplus_{i=1}^t M_i$, unique up to permutation, such that each $M_i$ is indecomposable.*

(iv) *$\mathrm{End}_R M$ is semiperfect and $\pi_\infty$-regular.*

PROOF. (i) Remark 4 shows $M = fM + \ker f$, so we need only show $fM \cap \ker f = 0$; a fortiori it suffices to show for some $t$ that $0 = \ker f^t \cap fM = \ker f^t \cap f^tM$. As in [**6**, p. 77], take a projective cover $\pi\colon P \to M$ (by assumption $P$ and $\ker \pi$ are f.g.) and also there is $g\colon P \to P$ such that $\pi g = f\pi$, i.e., $g$ completes the diagram

$$\begin{array}{ccccc} & & P & & \\ & \overset{g}{\swarrow} & \downarrow f\pi & & \\ P & \underset{\pi}{\rightarrow} & M & \rightarrow & 0 \end{array}$$

Clearly $g(\ker \pi) \subseteq \ker \pi$ (for if $\pi x = 0$, then $\pi(gx) = f\pi x = 0$), so Proposition 7 shows for large enough $t$ that $g^t(\ker \pi) = g^{t+1}(\ker \pi)$ and $g^t P = g^{t+1} P$. Proposition 6 implies $P = g^t P \oplus \ker g^t$, and $g^t$ restricts to an isomorphism from $g^t P$ to $g^t P$. Note $\pi g^t = f^t \pi$ by iteration.

Let $\pi'$ be the restriction of $\pi$ to $g^t P$. By Remark 5 we have a projective cover $\pi' \colon g^t P \to \pi' g^t P = f^t \pi P = f^t M$. Moreover $g^t(\ker \pi) \leq g^t P \cap \ker \pi \leq \ker \pi'$ implying

$$g^t(\ker \pi') \leq g^t(\ker \pi) = g^{2t}(\ker \pi) \leq g^t(\ker \pi'),$$

so equality holds at each step.

$g^{2t}(\ker \pi) = g^t(\ker \pi')$, so $g^t(\ker \pi) = \ker \pi'$. Thus $\ker \pi'$ is an f.g. module, and $f^t M$ is finitely presented.

Replacing $M, P, \pi$ respectively by $f^t M, g^t P$, and $\pi'$, we may assume $f$ is an onto map, $\pi = \pi'$ (so $g(\ker \pi) = \ker \pi'$) and $g$ is an isomorphism. But then $g$ restricts to a monomorphism from $\ker \pi$ to itself, which is thus an isomorphism by [**2**, Theorem 1.1]. Hence $\ker f = 0$, so $f$ is an isomorphism, as desired.

(ii) This is standard, by (i) and Proposition 7.

(iii) The decomposition exists by Proposition 1; each $M_i$ has a local endomorphism ring by (ii), so the Krull-Schmidt theorem (Azumaya's formulation) shows the decomposition is unique.

(iv) Let $E = \mathrm{End}_R M$. By (iii) $M = \bigoplus M_i$ so letting $e_i$ be the projection from $E$ to $M_i$, we see the $e_i$ are a complete set of orthogonal primitive idempotents, and $e_i E e_i \approx \mathrm{End}_R M_i$ is local by (ii), so $E$ is semiperfect by [**1**, Corollary 27.7]. $E$ is $\pi_\infty$-regular by [**2**, Proposition 2.3] applied to direct sums of copies of $M$. Q.E.D.

In order to apply this theorem we present a variant of (iv).

PROPOSITION 9. *Suppose $R$ is a semiperfect ring whose Jacobson radical $J$ is nil. If $M$ is an f.g. $R$-module and $E = \mathrm{End}_R M$, we have $\mathrm{Jac}(E)^k M \subseteq JM$ for some $k$.*

PROOF. Let $E' = \mathrm{End}_R(M/JM)$. $E'$ is semisimple Artinian by Morita theory, since $M/JM$ is f.g. over the semisimple Artinian ring $R/J$. There is a ring homomorphism $E \to E'$ given by $f \to \bar{f}$ where $\bar{f}(x + JM) = fx + JM$. The image $J'$ of $\mathrm{Jac}(E)$ is a nil subring (without 1) which is thus nilpotent. Hence $(J')^k = 0$ for some $k$, so $\mathrm{Jac}(E)^k M \subseteq JM$. Q.E.D.

COROLLARY 10. *Using notation as in Proposition 9, suppose $M$ is spanned by $n$ elements.*

(i) *If $J$ is locally nilpotent, then $\mathrm{Jac}(E)$ is locally nilpotent.*

(ii) *If $M_n(J)$ is left $T$-nilpotent, then $\mathrm{Jac}(E)$ is right $T$-nilpotent.*

(iii) *If $M_n(J)$ is right $T$-nilpotent, then $\mathrm{Jac}(E)$ is left $T$-nilpotent.*

(iv) *If $J$ is nilpotent, then $\mathrm{Jac}(E)$ is nilpotent.*

PROOF. (i) Let $S$ be a finite subset of $\mathrm{Jac}(E)$, and write $M = \sum_{i=1}^n Rx_i$. For $k$ as in the proposition we have $\{sx_i \colon s \in S^k,\ 1 \leq i \leq n\} \subseteq \sum_{i=1}^n J_0 x_i$ for some finite subset $J_0$ of $J$, and thus $J_0^q = 0$ for some $q$; hence $S^{kq} x_i = 0$ for all $i$, so $S^{kq} = 0$.

(ii), (iii) Passing to $M_n(R)$ and $M^{(n)}$ instead of $R$ and $M$ we have the same endomorphism ring $E$, so we may assume $M$ is cyclic (cf. proof of Proposition 7), i.e., $M = Rx$. Let $f_1, f_2, \dots$ be any sequence of elements of $\mathrm{Jac}(E)$. Then

$(f_1 \cdots f_k)x = a_1 x$ for some $a_1$ in $J$, and in general $f_{kt+1} \cdots f_{k(t+1)}x = a_{t+1}x$ for $a_{t+1}$ in $J$, yielding

$$f_1 \cdots f_{k(t+1)}x = f_1 \cdots f_{kt}a_{t+1}x = a_{t+1}f_1 \cdots f_{kt}x = \cdots = a_{t+1} \cdots a_1 x.$$

If $J$ is left (resp. right) $T$-nilpotent we thereby see $\mathrm{Jac}(E)$ is right (resp. left) $T$-nilpotent, as desired.

(iv) As in (ii) and (iii), noting that $J$ is nilpotent implies $M_n(J)$ is nilpotent.

COROLLARY 11. (Compare with Bjork [**3**, Theorems 4.1 and 4.2].) *Suppose $M$ is a finitely presented $R$-module, and $E = \mathrm{End}_R M$. If $R$ is left perfect, then $E$ is right perfect; if $R$ is right perfect, then $E$ is left perfect; if $R$ is semiprimary, then $E$ is semiprimary.*

PROOF. Combine Theorem 8 and Corollary 10, since these rings all are semiperfect and $\pi_\infty$-regular. Q.E.D.

**Discussion of results.** In this paper we have gone the route of Fitting's lemma, which holds by [**2**, Proposition 2.3] iff $\mathrm{End}_R M$ is left $\pi$-regular. Taking $M = R^{(n)}$ for each $n$, we see a necessary condition for Fitting's lemma to hold is for $R$ to be $\pi_\infty$-regular. On the other hand if we want a finitely presented module $M$ to be the direct sum of modules having local endomorphism rings, then in particular taking $M = R$ we see $R$ must be semiperfect (cf. [**1**, Corollary 27.7]). Thus the hypotheses of Theorem 8 are necessary for us to develop a Krull-Schmidt theory via Fitting's lemma and local endomorphism rings, and in this sense Theorem 8 is as strong as possible. However we have bypassed the question of classifying semiperfect $\pi_\infty$-regular rings in more intrinsic terms, such as the Jacobson radical. We shall address this question in the appendix and in [**10**].

On the other hand there are instances where a Krull-Schmidt theory can be obtained *without* Fitting's lemma. (For example if $P$ is an f.g. projective module over a semiperfect ring $R$, then $\mathrm{End}_R P$ is semiperfect by [**1**, Corollary 27.8].) In [**9**] we shall take up the question of what conditions on $R$ guarantee this for arbitrary finitely presented modules $P$.

Note that Proposition 9 also implies that if $M$ is spanned by $n$ elements and $R$ is semiperfect with $M_n(\mathrm{Jac}(R))$ nil, then $\mathrm{Jac}(\mathrm{End}_R M)$ is nil. (Proof: Pass to the cyclic case as usual, and writing $M = Rx$ and $f^k x = ax$ for $f$ in $\mathrm{Jac}(\mathrm{End}_R M)$ and $a$ in $J$, note $a^m = 0$ for some $m$, implying $f^{km} = 0$.)

Corollary 10 also shows in general that every nil subring of $\mathrm{End}_R M$ is nilpotent, whenever $M$ is an f.g. module over a semiprimary ring $R$.

Another kind of ring arising in these considerations is a ring $R$ which satisfies DCC on chains of principal left ideals of the form $Rs_1 > Rs_2s_1 > Rs_3s_2s_1 > \cdots$ whenever all $s_i$ are from a *finite* set $S$. It is easy to see that $\mathrm{Jac}(R)$ is then locally nilpotent. (Proof: Suppose $S$ is a finite subset of $\mathrm{Jac}(R)$ which is not nilpotent. Then there is $s_1$ in $S$ such that $S^k s_1 \neq 0$ for all $k$, for otherwise if $S^{k(s)}s = 0$, then $\max(k(s)) + 1$ would be a bound for the index of nilpotence of $S$. Continuing in this way one finds $s_2$ with $S^k s_2 s_1 \neq 0$ for all $k$, and so on, and clearly $Rs_1 > Rs_2s_1 > Rs_3s_2s_1 > \cdots$.) This seems to be a natural class of rings generalizing left perfect rings, so it would be nice to characterize them in terms of the Jacobson radical.

**Appendix: Examples of semiperfect left $\pi$-regular rings.** The results of this paper apply to the class of semiperfect $\pi_\infty$-regular rings. This leads one to search for classes of examples, particularly ones which are not perfect. As we observed earlier, any such ring $R$ has $\mathrm{Jac}(R)$ nil. On the other hand if $R$ is semilocal and $\mathrm{Jac}(R)$ is nil, then $R$ is semiperfect (since nil ideals are idempotent-lifting), so we would like to be able to conclude that $R$ is $\pi_\infty$-regular. Unfortunately this need not be true. In this appendix we show a semilocal ring $R$ is $\pi_\infty$-regular when $\mathrm{Jac}(R)$ is the *lower* nilradical of $R$, in particular when $R$ is a PI-ring (i.e., a ring satisfying a polynomial identity).

Our method of approach is to see what conclusions can be drawn from the existence of a ring $R$ such that $J = \mathrm{Jac}(R)$ is nil and $R/J$ is semisimple Artinian, but $R$ is *not* left $\pi$-regular; we shall call such $R$ a *counterexample.*

LEMMA 12. *If $R$ is a counterexample, then some prime homomorphic image of $R$ is a counterexample.*

PROOF. We rely on an elegant result of [**7**, Theorem 2.1], in which it is shown some prime homomorphic $R/P$ image of $R$ is not left $\pi$-regular. Let $J = \mathrm{Jac}(R) = \bigcap_{i=1}^n M_i$ for maximal ideals $M_i$ of $R$. (By hypothesis $J$ is nil and $R/J$ is semisimple Artinian, so $n$ is finite). Note that $M_1, \dots, M_n$ are the only prime ideals of $R$ containing $J$. Reordering the $M_i$ we may assume $P \subset M_i$ for $1 \le i \le t$, suitable $t$.

Let $\mathrm{Nil}(R/P) = N/P$, for suitable $N \triangleleft R$. For any $a \notin N$ there is some ideal $P_a \supset P$ of $R$ maximal with respect to missing all powers of $a$. It is standard that $R/P_a$ is prime with nilradical 0, so $J \subset P_a$, implying $P_a = M_i$ for suitable $1 \le i \le t$. Thus $\bigcap_{i=1}^t M_i \subseteq N$, and clearly $N \subseteq M_i$ for each $1 \le i \le t$, so $(R/P)/(N/P) \approx R/\bigcap_{i=1}^t M_i \approx \prod_{i=1}^t R/M_i$ is semisimple Artinian, as desired. Q.E.D.

(Lemma 12 is very closely related to [**5**, Proposition 2].)

PROPOSITION 13. *Suppose $R$ is semilocal and* $\mathrm{Jac}(R)$ *equals the lower nilradical $L(R)$ of $R$. Then $R$ is $\pi_\infty$-regular.*

PROOF. First note that $R/L(R)$ is semisimple Artinian and thus left $\pi$-regular. Thus each prime image of $R$ is left $\pi$-regular, so $R$ is left $\pi$-regular by Lemma 12. Furthermore the hypotheses pass to $M_n(R)$ for each $n$; $\mathrm{Jac}(M_n(R))$ is the lower nilradical of $M_n(R)$ (seen by viewing the lower nilradical as the intersection of all prime ideals), and $M_n(R)$ is semilocal, implying $M_n(R)$ is left $\pi$-regular. Hence $R$ is $\pi_\infty$-regular. Q.E.D.

COROLLARY 14. *If $R$ is a semilocal PI-ring and* $\mathrm{Jac}(R)$ *is nil, then $R$ is $\pi_\infty$-regular.*

PROOF. By [**8**, Theorem 1.6.36] any nil ideal is in the lower nilradical. Q.E.D.

As stated earlier, there *is* a counterexample, which also provides a left $\pi$-regular ring which is not $\pi_\infty$-regular, thereby answering a long-standing question in the theory of $\pi$-regular rings. Since the construction is rather intricate, it will appear separately in [**10**].

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