

## ON CERTAIN GROUPS OF CENTRAL TYPE

ALBERTO ESPUELAS

ABSTRACT. A finite group  $G$  is a group of central type if there exists  $\chi \in \text{Irr}(G)$  with  $\chi(1)^2 = |G:Z(G)|$ . It is known that, in such conditions,  $G$  is solvable. Here some conditions assuring the nilpotence of groups of central type are given.

Let  $G$  be a finite group,  $N$  a normal subgroup of  $G$  and  $\theta \in \text{Irr}(N)$  invariant in  $G$ . Then we say that  $(G, N, \theta)$  is a character triple. Such a triple is *fully ramified* provided that one of the following (equivalent) conditions holds:

- (i) If  $\chi \in \text{Irr}(G|\theta)$  then  $\chi$  vanishes off  $N$ .
  - (ii)  $\theta^G$  has a unique irreducible constituent  $\chi$ .
  - (iii) If  $\chi \in \text{Irr}(G|\theta)$  then  $(\chi(1)/\theta(1))^2 = |G:N|$
- (see [5, p. 95]).

We say that  $\chi$  is fully ramified in  $N$  and  $\theta$  is fully ramified in  $G$ . A group  $G$  is of *central type* provided that there exists  $\chi \in \text{Irr}(G)$  with  $\chi(1)^2 = |G:Z(G)|$ . If  $\lambda$  is the irreducible constituent of  $\chi_{Z(G)}$  then  $(G, Z(G), \lambda)$  is fully ramified. It is well known (see Lemma 4.3 of [3]) that if  $(G, N, \theta)$  is fully ramified then there exists a group  $G_1$  of central type and  $\lambda \in \text{Irr}(Z(G_1))$  such that  $(G, N, \theta)$  and  $(G_1, Z(G_1), \lambda)$  are isomorphic character triples (see [5, p. 187] for the definition).

In [3] Howlett and Isaacs introduced the concept of an irreducible group of central type. We recall that definition: Let  $G$  be a group of central type with  $\lambda \in \text{Irr}(Z(G))$  fully ramified in  $G$ . Then  $G$  is *reducible* if  $\lambda$  is fully ramified in some normal subgroup  $N$  of  $G$  with  $Z(G) < N < G$ . Otherwise,  $G$  is *irreducible*.

In [2] Gagola raised the question about the nilpotence of irreducible groups of central type. The answer is negative as Example 8.2 of [3] shows. Our result is the following

**THEOREM.** *Let  $G$  be an irreducible group of central type,  $Z = Z(G)$  and  $\bar{G} = G/Z$ . Let  $\lambda \in \text{Irr}(Z(G))$  be fully ramified in  $G$ . Then*

(i) *If there exists a prime  $p$  such that  $|\text{socle}(\bar{G})|_p > |\bar{G}:\text{socle}(\bar{G})|_p$  then  $G$  is nilpotent.*

(ii)  *$F(\bar{G}/\text{socle}(\bar{G})) = F(\bar{G})/\text{socle}(\bar{G})$ , where  $F(X)$  denotes the Fitting subgroup of  $X$ . As a consequence, if  $\bar{G}/\text{socle}(\bar{G})$  has a (nontrivial) normal Sylow subgroup then  $G$  is nilpotent.*

We need the following

**LEMMA.** *Let  $G$  be an irreducible group of central type,  $Z = Z(G)$  and  $\bar{G} = G/Z$ . Let  $\lambda \in \text{Irr}(Z(G))$  be fully ramified in  $G$ . Then, assuming that  $G$  is not nilpotent, we have that  $\lambda$  is extendible to the preimage of  $\text{socle}(\bar{G})$  in  $G$ .*

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Received by the editors December 14, 1984 and, in revised form, May 1, 1985.  
1980 *Mathematics Subject Classification.* Primary 20C15.

PROOF OF THE LEMMA. The group  $G$  is solvable by Theorem 7.3 of [3]. We may assume that  $\lambda$  is faithful.

First we show that  $\lambda$  is extendible to the preimage  $A$  in  $G$  of any minimal normal subgroup  $\bar{A}$  of  $\bar{G}$ . As  $\bar{G}$  is solvable,  $\bar{A}$  is an elementary abelian group. Then there exists a subgroup  $A_1$  normal in  $G$  with  $Z \leq A_1 \leq A$  such that  $\lambda$  is extendible to  $A_1$  and  $(A, A_1, \psi)$  is fully ramified for any  $\psi \in \text{Irr}(A_1|\lambda)$  (see [4, Theorem 2.7]). As  $G$  is irreducible and  $\bar{A}$  is a chief factor of  $\bar{G}$ , we have  $A = A_1$  and our claim is verified.

Let  $X$  be the preimage in  $G$  of  $\text{socle}(\bar{G})$ . Take  $\bar{A}_1$  and  $\bar{A}_2$  minimal normal subgroups of  $G$  such that  $|\bar{A}_1| \leq |\bar{A}_2|$ . Let  $\chi$  be the unique irreducible constituent of  $\lambda^G$ . If  $\psi$  is an extension of  $\lambda$  to  $A_1$ , then  $\text{Irr}(A_1|\lambda) = \{\psi\mu | \psi \in \text{Irr}(A_1/Z)\}$ . But  $\text{Irr}(A_1|\lambda)$  is also the set of irreducible constituents of  $\chi_{A_1}$  and hence  $G$  permutes its elements transitively by conjugation.

As  $\text{socle}(\bar{G})$  is abelian then  $I_{A_2}(\psi) = I_{A_2}(\psi\mu)$  for every  $\mu$  and hence  $I_{A_2}(\psi)$  is normal in  $G$ . Thus  $I_{A_2}(\psi) = A_2$  or  $I_{A_2}(\psi) = Z$ . In the second case the character  $\psi^{A_1A_2}$  is irreducible and its degree is  $|\bar{A}_2|$ . Hence  $|\bar{A}_2|^2 \leq |A_1A_2 : Z(A_1A_2)|$ . As  $Z$  is central in  $A_1A_2$  this forces to  $|\bar{A}_1| = |\bar{A}_2|$  and thus  $(A_1A_2, Z, \lambda)$  is fully ramified. As  $A_1A_2$  is a normal nilpotent subgroup of  $G$  this contradicts the irreducibility of  $G$ . Hence  $A_2$  leaves invariant each  $\psi\mu$  and, as  $\psi\mu$  is linear, we have

$$[A_1, A_2] \leq \bigcap_{\mu} \text{Ker}(\psi\mu) = \text{Ker} \lambda^{A_1} = 1$$

since  $\lambda$  is faithful.

Now  $X$  is abelian and clearly  $\lambda$  is extendible to  $X$ .

PROOF OF THE THEOREM. As in the Lemma,  $G$  is solvable and  $\lambda$  is faithful.

(i) Let  $X$  be the preimage of  $\text{socle}(\bar{G})$  in  $G$ . If  $G$  is not nilpotent then  $\lambda$  is extendible to  $X$  by the Lemma. Take a prime  $p$  satisfying the hypothesis and let  $P \in S_p(G)$ . Put  $\lambda = \prod_q \lambda_q$  where  $\lambda_q \in \text{Irr}(Z_q)$ ,  $Z_q \in S_q(Z)$ . Now  $(P, Z(P), \lambda_p)$  is fully ramified and  $Z(P) = Z \cap P$  (see [1, Theorem 2]). Let  $\chi_p$  be the unique irreducible constituent of  $\lambda_p^P$ . Let  $\psi_p$  be an extension of  $\lambda_p$  to  $X \cap P$ . Now  $\chi_p$  is a constituent of  $\psi_p^P$  and  $\chi_p(1)^2 \leq |P : X \cap P|^2 = |\bar{G} : \text{socle}(\bar{G})|_p^2 < |\bar{G}|_p = |P : Z(P)|$ , a contradiction.

(ii) Put  $\lambda = \prod_q \lambda_q$ , where  $\lambda_q \in \text{Irr}(Z_q)$ ,  $Z_q \in S_q(Z)$ . If  $Q \in S_q(G)$  then  $Z(Q) = Q \cap Z$  and  $(Q, Z(Q), \lambda_q)$  is fully ramified as in (i). Let  $\bar{X}_q \in S_q(\bar{X})$ ,  $q$  a prime. If  $\chi_q$  is the unique irreducible constituent of  $\lambda_q^Q$  then the elements of  $\text{Irr}(X_q \cap Q|\lambda_q)$  are the irreducible constituents of  $(\chi_q)_{X_q \cap Q}$ . Hence they are conjugate in  $Q$ . But  $\lambda_q$  is invariant in  $G$  and  $X_q \cap Q$  is normal in  $G$ . Hence  $G$  permutes by conjugation the elements of  $\text{Irr}(X_q \cap Q|\lambda_q)$ . We conclude that if  $\psi_q \in \text{Irr}(X_q \cap Q|\lambda_q)$  then  $I_G(\psi_q)$  contains a Hall  $q'$ -subgroup of  $G$ .

We may assume that  $G$  is not nilpotent and then the Lemma applies. Hence the elements of  $\text{Irr}(X_q \cap Q|\lambda_q)$  are linear and  $X$  is abelian as we showed. Now let  $T$  be a subgroup of  $G$  such that  $\bar{T}/\bar{X}$  is a normal  $p$ -subgroup of  $\bar{G}/\bar{X}$  for a prime  $p$ . We show that  $O_{p'}(\bar{X}) \leq Z(\bar{T})$ . Take  $q \neq p$  and  $\bar{X}_q \in S_q(\bar{X})$ . As  $X$  is abelian and  $T/X$  is a normal  $q'$ -subgroup of  $G/X$ , we have that  $T$  leaves invariant each element of  $\text{Irr}(X_q \cap Q|\lambda_q)$  by the preceding paragraph. Thus  $[T, X_q \cap Q] \leq \text{Ker} \lambda_q^{X_q \cap Q} = 1$  since  $\lambda_q$  is faithful. Thus our claim is verified and hence  $\bar{T}$  is nilpotent. Then  $F(\bar{G}/\text{socle}(\bar{G})) \leq F(\bar{G})/\text{socle}(\bar{G})$ . As the reverse inclusion is obvious, the first part

of (ii) is verified. To prove the second take a prime  $p$  and that  $\overline{G}/\text{socle}(\overline{G})$  is  $p$ -closed and let  $P$  be a Sylow  $p$ -subgroup of  $G$ . Now  $P$  is normal in  $G$ . As  $(PZ, Z, \lambda)$  is fully ramified, we deduce that  $G$  is nilpotent.

We recall the following definition (see [2, p. 123]): A  $p$ -group  $Q$  is *reductive* if, for every fully ramified triple  $(H, Z, \lambda)$  with  $Q$  isomorphic to a Sylow  $p$ -subgroup of  $H/Z$ , the irreducible constituent  $\chi$  of  $\lambda^H$  is fully ramified in  $O^p(H)Z$ .

In Lemma 4.4 of [2] Gagola showed that every  $p$ -group of order  $p^4$  is reductive. There exist nonreductive  $p$ -groups at least for  $p = 2, 3$ , as Example 8.2 of [3] shows. As this example is really complicated and it is not apparent that it might be extended to the remaining primes, we construct a nonreductive  $p$ -group of order  $p^6$  for  $p$  odd.

We consider  $A_0 = \langle a, b, c, z \rangle \simeq (C_p)^4$ . By the theory of cyclic extensions there exists an extension  $A$  of  $A_0$  by  $B_0 = \langle \alpha, \beta, \gamma \rangle \simeq (C_p)^3$  where  $\alpha^p = \beta^p = \gamma^p = 1$ ,  $[\alpha, \beta] = c$ ,  $[\alpha, \gamma] = a$ ,  $[\beta, \gamma] = b$ ,  $[a, \alpha] = [b, \beta]^{-1} = [c, \gamma] = z$ , the remaining commutators being trivial.

Let  $B$  be the quaternion group of order 8 acting on  $A$  as follows: If  $e$  and  $f$  are generators of  $B$  then  $\alpha^f = \beta$ ,  $\beta^f = \alpha$ ,  $\gamma^f = \gamma^{-1}$ ,  $a^f = b^{-1}$ ,  $b^f = a^{-1}$ ,  $c^f = c^{-1}$ ,  $z^f = z$  and  $e$  acts trivially.

It is an easy check that this action is well defined. Let  $G$  be the natural semidirect product of  $A$  by  $B$ . Let  $\lambda$  be a nonprincipal character of  $A_0/\langle a, b, c \rangle$  and  $\mu$  a faithful character of  $\langle e \rangle$ . Consider  $\lambda \times \mu$  as a character of  $A_0 \times \langle e \rangle$ . Clearly  $I_G(\lambda \times \mu) = I_G(\lambda) \cap I_G(\mu)$ . It is an easy check that  $N_A(\text{Ker } \lambda) = A_0$  and hence  $I_A(\lambda) = A_0$ . Clearly  $I_B(\mu) = \langle e \rangle$ . Then  $I_G(\lambda \times \mu) = A_0 \times \langle e \rangle$ . Thus  $(\lambda \times \mu)^G$  is irreducible and  $(\lambda \times \mu)^G(1)^2 = |G : Z(G)|$ . Hence  $G$  is of central type. Now  $O^p(G)$  contains  $\alpha^{-1}\beta, \gamma$  and  $c$ . As is normal in  $G$ , it contains  $[\alpha, \gamma]$  and  $[\beta, \gamma]$ . Thus  $|G : O^p(G)Z| = p$  and  $(\lambda \times \mu)^G$  is not fully ramified in  $O^p(G)Z$ .

Using a similar technique it is possible to construct a nonreductive 2-group of order  $2^6$ .

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DEPARTAMENTO DE ALGEBRA, FACULTAD DE CIENCIAS, UNIVERSIDAD DE ZARAGOZA,  
50009 ZARAGOZA, SPAIN