PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 97, Number 1, May 1986

ON CERTAIN GROUPS OF CENTRAL TYPE

ALBERTO ESPUELAS

ABSTRACT. A finite group G is a group of central type if there exists $\chi \in \operatorname{Irr}(G)$ with $\chi(1)^2 = |G\colon Z(G)|$. It is known that, in such conditions, G is solvable. Here some conditions assuring the nilpotence of groups of central type are given.

Let G be a finite group, N a normal subgroup of G and $\theta \in \operatorname{Irr}(N)$ invariant in G. Then we say that (G, N, θ) is a character triple. Such a triple is *fully ramified* provided that one of the following (equivalent) conditions holds:

(i) If $\chi \in \operatorname{Irr}(G|\theta)$ then χ vanishes off N.

(ii) θ^G has a unique irreducible constituent χ.

(iii) If $\chi \in \operatorname{Irr}(G|\theta)$ then $(\chi(1)/\theta(1))^2 = |G\colon N|$

(see [**5**, p. 95]).

We say that χ is fully ramified in N and θ is fully ramified in G. A group G is of *central type* provided that there exists $\chi \in \operatorname{Irr}(G)$ with $\chi(1)^2 = |G\colon Z(G)|$. If λ is the irreducible constituent of $\chi_{Z(G)}$ then $(G, Z(G), \lambda)$ is fully ramified. It is well known (see Lemma 4.3 of [**3**]) that if (G, N, θ) is fully ramified then there exists a group G_1 of central type and $\lambda \in \operatorname{Irr}(Z(G_1))$ such that (G, N, θ) and $(G_1, Z(G_1), \lambda)$ are isomorphic character triples (see [**5**, p. 187] for the definition).

In [**3**] Howlett and Isaacs introduced the concept of an irreducible group of central type. We recall that definition: Let G be a group of central type with $\lambda \in \operatorname{Irr}(Z(G))$ fully ramified in G. Then G is *reducible* if λ is fully ramified in some normal subgroup N of G with $Z(G) < N < G$. Otherwise, G is *irreducible.*

In [**2**] Gagola raised the question about the nilpotence of irreducible groups of central type. The answer is negative as Example 8.2 of [**3**] shows. Our result is the following

THEOREM. *Let G be an irreducible group of central type, $Z = Z(G)$ and $\overline{G} = G/Z$. Let $\lambda \in \operatorname{Irr}(Z(G))$ be fully ramified in G. Then*

(i) *If there exists a prime p such that $|\operatorname{socle}(\overline{G})|_p > |\overline{G}\colon \operatorname{socle}(\overline{G})|_p$ then G is nilpotent.*

(ii) *$F(\overline{G}/\operatorname{socle}(\overline{G})) = F(\overline{G})/\operatorname{socle}(\overline{G})$, where $F(X)$ denotes the Fitting subgroup of X. As a consequence, if $\overline{G}/\operatorname{socle}(\overline{G})$ has a (nontrivial) normal Sylow subgroup then G is nilpotent.*

We need the following

LEMMA. *Let G be an irreducible group of central type, $Z = Z(G)$ and $\overline{G} = G/Z$. Let $\lambda \in \operatorname{Irr}(Z(G))$ be fully ramified in G. Then, assuming that G is not nilpotent, we have that λ is extendible to the preimage of $\operatorname{socle}(\overline{G})$ in G.*

Received by the editors December 14, 1984 and, in revised form, May 1, 1985.
1980 *Mathematics Subject Classification.* Primary 20C15.

PROOF OF THE LEMMA. The group G is solvable by Theorem 7.3 of [**3**]. We may assume that λ is faithful.

First we show that λ is extendible to the preimage A in G of any minimal normal subgroup $\overline{A}$ of $\overline{G}$. As $\overline{G}$ is solvable, $\overline{A}$ is an elementary abelian group. Then there exists a subgroup A_1 normal in G with $Z \leq A_1 \leq A$ such that λ is extendible to A_1 and (A, A_1, ψ) is fully ramified for any $\psi \in \mathrm{Irr}(A_1|\lambda)$ (see [**4**, Theorem 2.7]). As G is irreducible and $\overline{A}$ is a chief factor of $\overline{G}$, we have $A = A_1$ and our claim is verified.

Let X be the preimage in G of socle($\overline{G}$). Take $\overline{A}_1$ and $\overline{A}_2$ minimal normal subgroups of G such that $|\overline{A}_1| \leq |\overline{A}_2|$. Let χ be the unique irreducible constituent of λ^G. If ψ is an extension of λ to A_1, then $\mathrm{Irr}(A_1|\lambda) = \{\psi\mu | \psi \in \mathrm{Irr}(A_1/Z)\}$. But $\mathrm{Irr}(A_1|\lambda)$ is also the set of irreducible constituents of χ_{A_1} and hence G permutes its elements transitively by conjugation.

As socle($\overline{G}$) is abelian then $I_{A_2}(\psi) = I_{A_2}(\psi\mu)$ for every μ and hence $I_{A_2}(\psi)$ is normal in G. Thus $I_{A_2}(\psi) = A_2$ or $I_{A_2}(\psi) = Z$. In the second case the character $\psi^{A_1A_2}$ is irreducible and its degree is $|\overline{A}_2|$. Hence $|\overline{A}_2|^2 \leq |A_1A_2 : Z(A_1A_2)|$. As Z is central in A_1A_2 this forces to $|\overline{A}_1| = |\overline{A}_2|$ and thus (A_1A_2, Z, λ) is fully ramified. As A_1A_2 is a normal nilpotent subgroup of G this contradicts the irreducibility of G. Hence A_2 leaves invariant each $\psi\mu$ and, as $\psi\mu$ is linear, we have

$$[A_1, A_2] \leq \bigcap_{\mu} \mathrm{Ker}(\psi\mu) = \mathrm{Ker}\,\lambda^{A_1} = 1$$

since λ is faithful.

Now X is abelian and clearly λ is extendible to X.

PROOF OF THE THEOREM. As in the Lemma, G is solvable and λ is faithful.

(i) Let X be the preimage of socle($\overline{G}$) in G. If G is not nilpotent then λ is extendible to X by the Lemma. Take a prime p satisfying the hypothesis and let $P \in S_p(G)$. Put $\lambda = \prod_q \lambda_q$ where $\lambda_q \in \mathrm{Irr}(Z_q)$, $Z_q \in S_q(Z)$. Now $(P, Z(P), \lambda_p)$ is fully ramified and $Z(P) = Z \cap P$ (see [**1**, Theorem 2]). Let χ_p be the unique irreducible constituent of λ_p^P. Let ψ_p be an extension of λ_p to $X \cap P$. Now χ_p is a constituent of ψ_p^P and $\chi_p(1)^2 \leq |P : X \cap P|^2 = |\overline{G} : \mathrm{socle}(\overline{G})|_p^2 < |\overline{G}|_p = |P : Z(P)|$, a contradiction.

(ii) Put $\lambda = \prod_q \lambda_q$, where $\lambda_q \in \mathrm{Irr}(Z_q)$, $Z_q \in S_q(Z)$. If $Q \in S_q(G)$ then $Z(Q) = Q \cap Z$ and $(Q, Z(Q), \lambda_q)$ is fully ramified as in (i). Let $\overline{X}_q \in S_q(\overline{X})$, q a prime. If χ_q is the unique irreducible constituent of λ_q^Q then the elements of $\mathrm{Irr}(X_q \cap Q|\lambda_q)$ are the irreducible constituents of $(\chi_q)_{X_q \cap Q}$. Hence they are conjugate in Q. But λ_q is invariant in G and $X_q \cap Q$ is normal in G. Hence G permutes by conjugation the elements of $\mathrm{Irr}(X_q \cap Q|\lambda_q)$. We conclude that if $\psi_q \in \mathrm{Irr}(X_q \cap Q|\lambda_q)$ then $I_G(\psi_q)$ contains a Hall q'-subgroup of G.

We may assume that G is not nilpotent and then the Lemma applies. Hence the elements of $\mathrm{Irr}(X_q \cap Q|\lambda_q)$ are linear and X is abelian as we showed. Now let T be a subgroup of G such that $\overline{T}/\overline{X}$ is a normal p-subgroup of $\overline{G}/\overline{X}$ for a prime p. We show that $O_{p'}(\overline{X}) \leq Z(\overline{T})$. Take $q \neq p$ and $\overline{X}_q \in S_q(\overline{X})$. As X is abelian and T/X is a normal q'-subgroup of G/X, we have that T leaves invariant each element of $\mathrm{Irr}(X_q \cap Q|\lambda_q)$ by the preceding paragraph. Thus $[T, X_q \cap Q] \leq \mathrm{Ker}\,\lambda_q^{X_q \cap Q} = 1$ since λ_q is faithful. Thus our claim is verified and hence $\overline{T}$ is nilpotent. Then $F(\overline{G}/\mathrm{socle}(\overline{G})) \leq F(\overline{G})/\mathrm{socle}(\overline{G})$. As the reverse inclusion is obvious, the first part of (ii) is verified. To prove the second take a prime p and that $\overline{G}/\mathrm{socle}(\overline{G})$ is p-closed and let P be a Sylow p-subgroup of G. Now P is normal in G. As (PZ, Z, λ) is fully ramified, we deduce that G is nilpotent.

We recall the following definition (see [**2**, p. 123]): A p-group Q is *reductive* if, for every fully ramified triple (H, Z, λ) with Q isomorphic to a Sylow p-subgroup of H/Z, the irreducible constituent χ of λ^H is fully ramified in $O^p(H)Z$.

In Lemma 4.4 of [**2**] Gagola showed that every p-group of order p^4 is reductive. There exist nonreductive p-groups at least for $p = 2, 3$, as Example 8.2 of [**3**] shows. As this example is really complicated and it is not apparent that it might be extended to the remaining primes, we construct a nonreductive p-group of order p^6 for p odd.

We consider $A_0 = \langle a, b, c, z\rangle \simeq (C_p)^4$. By the theory of cyclic extensions there exists an extension A of A_0 by $B_0 = \langle \alpha, \beta, \gamma\rangle \simeq (C_p)^3$ where $\alpha^p = \beta^p = \gamma^p = 1$, $[\alpha, \beta] = c$, $[\alpha, \gamma] = a$, $[\beta, \gamma] = b$, $[a, \alpha] = [b, \beta]^{-1} = [c, \gamma] = z$, the remaining commutators being trivial.

Let B be the quaternion group of order 8 acting on A as follows: If e and f are generators of B then $\alpha^f = \beta$, $\beta^f = \alpha$, $\gamma^f = \gamma^{-1}$, $a^f = b^{-1}$, $b^f = a^{-1}$, $c^f = c^{-1}$, $z^f = z$ and e acts trivially.

It is an easy check that this action is well defined. Let G be the natural semidirect product of A by B. Let λ be a nonprincipal character of $A_0/\langle a, b, c\rangle$ and μ a faithful character of $\langle e\rangle$. Consider $\lambda \times \mu$ as a character of $A_0 \times \langle e\rangle$. Clearly $I_G(\lambda \times \mu) = I_G(\lambda) \cap I_G(\mu)$. It is an easy check that $N_A(\mathrm{Ker}\,\lambda) = A_0$ and hence $I_A(\lambda) = A_0$. Clearly $I_B(\mu) = \langle e\rangle$. Then $I_G(\lambda \times \mu) = A_0 \times \langle e\rangle$. Thus $(\lambda \times \mu)^G$ is irreducible and $(\lambda \times \mu)^G(1)^2 = |G : Z(G)|$. Hence G is of central type. Now $O^p(G)$ contains $\alpha^{-1}\beta, \gamma$ and c. As is normal in G, it contains $[\alpha, \gamma]$ and $[\beta, \gamma]$. Thus $|G : O^p(G)Z| = p$ and $(\lambda \times \mu)^G$ is not fully ramified in $O^p(G)Z$.

Using a similar technique it is possible to construct a nonreductive 2-group of order 2^6.

References

1. F. De Meyer and G. J. Janusz, *Finite groups with an irreducible representation of large degree*, Math. Z. **108** (1969), 145–153.
2. S. M. Gagola, *Characters fully ramified over a normal subgroup*, Pacific J. Math. **55** (1974), 107–126.
3. R. B. Howlett and I. M. Isaacs, *On groups of central type*, Math. Z. **179** (1982), 555–569.
4. I. M. Isaacs, *Characters of solvable and symplectic groups*, Amer. J. Math. **95** (1973), 594–635.
5. ——, *Character theory of finite groups*, Academic Press, New York, 1976.

Departamento de Algebra, Facultad de Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain