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A NOTE ON A THEOREM OF PERRON

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ABSTRACT. Given an infinite simple continued fraction $[a_0, a_1, \ldots, a_n, \ldots]$, let M_n denote $[0, a_n, a_{n-1}, \ldots, a_1] + [a_{n+1}, a_{n+2}, \ldots]$. A well-known result due to Perron [**1**, III, 212] states: If $a_{n+2} = m$, then there is a k in $\{n, n+1, n+2\}$ for which $M_k > \sqrt{m^2+4}$. In this note we give a new proof for this result and add that there is a j in $\{n, n+1, n+2\}$ for which $M_j < \sqrt{m^2+4}$.

1. Introduction. Let x be an irrational number whose simple continued fraction expansion is $[a_0, a_1, a_2, \ldots]$. It is well known [**1**, Chapter III, 1(6)] that $|x - p_n/q_n| = 1/M_n q_n^2$, where $p_n/q_n = [a_0, a_1, \ldots, a_n]$ and $M_n = [0, a_n, a_{n-1}, \ldots, a_1] + [a_{n+1}, a_{n+2}, \ldots]$. The following result, essentially due to Perron [**1**, III, 2.12], has received much attention:

(*) If $a_{n+2} = m$, then $M_k > \sqrt{m^2+4}$ for at least one value of k in $\{n, n+1, n+2\}$.

This note displays a simple strategy which leads to this result and also unfolds another interesting fact. We prove the following

THEOREM. *If $a_{n+2} = m$, then there is a k in $\{n, n+1, n+2\}$ for which $M_k > \sqrt{m^2+4}$, and there is a j in $\{n, n+1, n+2\}$ for which $M_j < \sqrt{m^2+4}$.*

2. Proof of the Theorem. Set $a_{n+2} = m$, $u = \sqrt{m^2+4}$, $s = [0, a_{n+3}, a_{n+4}, \ldots]$, and $t = [a_{n+1}, a_n, \ldots, a_1]$. Now $M_n = t + 1/(m+s)$, $M_{n+1} = 1/t + m + s$ and $M_{n+2} = t/(mt+1) + 1/s$. Again put $h_1(t) = 1/(u-t) - m$, $h_2(t) = u - m - 1/t$ and $h_3(t) = (mt+1)/((um-1)t+u)$. Also

$$h_2(t) - h_1(t) = -\frac{u}{t(u-t)}\left(t - \frac{u+m}{2}\right)\left(t - \frac{u-m}{2}\right)$$

and

$$h_3(t) - h_2(t) = -\frac{u(mu - m^2 - 1)}{t(mut - t + u)}\left(t - \frac{u+m}{2}\right)\left(t + \frac{3m+u}{2(2m^2-1)}\right).$$

Note that $(u-m)/2 < 1 \leq t$ and $u \neq t$ because t is rational and u is not. The following cases appear:

Case 1. $u > t$ and $t < (u+m)/2$. In this case

$$\operatorname{Sign}\{h_2(t) - h_1(t)\} = (-)\cdot(-)\cdot(+) = (+),$$
$$\operatorname{Sign}\{h_3(t) - h_2(t)\} = (-)\cdot(-)\cdot(+) = (+).$$

So $h_1(t) < h_2(t) < h_3(t)$. Thus if $s \leq h_1(t)$, then we have $s < h_2(t)$ and $s < h_3(t)$, which are equivalent to $M_{n+1} < u$ and $M_{n+2} > u$, respectively. Or if $h_1(t) < s \leq$

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$h_2(t)$, then we have $h_1(t) < s$ and $s < h_3(t)$, which are equivalent to $M_n < u$ and $M_{n+2} > u$, respectively. Or if $h_2(t) < s$, then we have $h_1(t) < s$ and $h_2(t) < s$, which are equivalent to $M_n < u$ and $M_{n+1} > u$, respectively.

Case 2. $u > t$ and $(u+m)/2 < t$. In this case we get $h_3(t) < h_2(t) < h_1(t)$ and argue as above.

Case 3. $u < t$. In this case $h_1(t) < 0 < s$ or $M_n > u$. Also $(u+m)/2 < u < t$. Therefore $\mathrm{Sign}\{h_3(t) - h_2(t)\} = (-)$. So if $s \leq h_3(t)$, we get $s < h_2(t)$ or $M_{n+1} < u$. Otherwise we have $h_3(t) < s$ or $M_{n+2} < u$.

This completes the proof of the theorem.

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