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ON A QUESTION OF FEIT

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ABSTRACT. The following theorem is proved: Assume χ is an irreducible complex character of the finite group G and G is π-solvable where π is the set of prime divisors of $\chi(1)$. Then G contains an element of order $f(\chi)$.

Introduction. All groups in this paper are finite. The question referred to in the title is the following: Let χ be an irreducible complex character of a finite group G and let $f(\chi)$ be the smallest positive integer such that $\{\chi(g)|g \in G\} \subseteq Q(\alpha)$ where α is a primitive $f(\chi)$th root of unity. Does G contain an element of order $f(\chi)$?

By using factorizations of quasi-primitive irreducible characters into products of characters we obtain the following result:

THEOREM. *Assume χ is an irreducible complex character of a finite group G and G is π-solvable where π is the set of primes dividing $\chi(1)$. Then G contains an element of order $f(\chi)$.*

This theorem yields the following Corollary which was proved independently by Amit and Chillag [**1**].

COROLLARY. *If χ is an irreducible character of a solvable group G, then G contains an element of order $f(\chi)$.*

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1. We introduce some definitions and notation. Let σ be a nonempty set of primes and h be an element in a group G, h is a σ-element if $\langle h \rangle$ is a σ-group. Any $g \in G$ may be written uniquely as $g = g_\sigma g_{\sigma'}$, where g_σ is a σ-element, $g_{\sigma'}$ is a σ'-element and both g_σ and $g_{\sigma'}$ are powers of g. The elements g_σ and $g_{\sigma'}$ are called the σ-part and σ'-part of g.

If Ω is a Galois extension of the field $\Omega_1, G(\Omega/\Omega_1)$ denotes the Galois group of Ω over Ω_1.

If $\chi \in \mathrm{Irr}(G)$ and σ is a set of primes, then χ is σ-special provided that $\chi(1)$ is a σ-number and that for all subnormal subgroups S of G and all irreducible constituents θ of χ_S, the determinantal order $O(\theta)$ is a σ-number.

PROOF OF THEOREM. The result is clear for linear characters so by induction on $\chi(1)$ we assume that the theorem is true for irreducible characters Φ and groups H satisfying the hypothesis if $\Phi(1) < \chi(1)$. If $\chi = \Phi^G$ where $\Phi \in \mathrm{Irr}(H)$ and

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$|H| < |G|$, then $f(\chi)|f(\Phi)$. Since $\Phi(1)$ is a proper divisor of $\chi(1)$, H contains an element of order $f(\Phi)$ and the theorem follows. Thus, we may assume that χ is primitive.

Let $f(\chi) = \prod_{i=1}^{n} p_i^{a_i}$ where $a_i \geq 1$ and the p_i are distinct primes. Let $p_i^{r_i}$ denote the order of a Sylow p_i-subgroup of G for $i = 1, \ldots, n$. Then Ω_i denotes the field of $(|G|/p_i^{r_i - a_i + 1})$th roots of unity over Q. Let Ω be a field of $|G|$th roots of unity over Q. By the definition of $f(\chi)$, we may choose a $\theta_i \in G(\Omega/\Omega_i)$ such that θ_i does not leave χ invariant for $i = 1, \ldots, n$. We first show that no product of an odd number of distinct elements in $\{\theta_1, \ldots, \theta_n\}$ leaves χ invariant. Assume otherwise; then since $G(\Omega/Q)$ is abelian, we may choose notation so that $\beta = \prod_{i=1}^{r} \theta_i$ leaves χ invariant, for some $r \geq 1$.

Suppose $p_i|\chi(1)$ for some $i = 1, \ldots, r$. Since $G(\Omega/Q)$ is abelian, we may assume $p_1|\chi(1)$. By [**4**, Corollary 2.7], $\chi = \chi_1\Phi$ where χ_1 and Φ are primitive irreducible p_1-special and p_1'-special characters. G is p_1-solvable, hence p_1'-separable, and so $f(\Phi)$ is a p_1'-number by [**3**, Proposition 6.3(a)].

If $\theta \in G(\Omega/Q)$ and σ is any set of primes, then it is direct to see that whenever γ is an irreducible σ-special character so is $\theta \circ \gamma$. Hence $\beta \circ \chi_1$ and $\beta \circ \Phi$ are p_1-special and p_1'-special. Now $\chi_1\Phi = \chi = \beta \circ \chi = (\beta \circ \chi_1)(\beta \circ \Phi)$ and [**4**, Theorem 2.2] yield $\chi_1 = \beta \circ \chi_1$. By [**3**, Proposition 6.3(a)], $f(\chi_1)$ is a p_1-number so $\theta_i \circ \chi_1 = \chi_1$ for $i = 2, \ldots, r$. Since $G(\Omega/Q)$ is abelian, it follows that $\chi_1 = \beta \circ \chi_1 = \theta_1 \circ \chi_1$. Now $f(\Phi)$ a p_1'-number implies that $\theta_1 \circ \Phi = \Phi$. However, $\theta_1 \circ \chi = (\theta_1 \circ \chi_1)(\theta_1 \circ \Phi) = \chi_1\Phi = \chi$ contradicts the choice of θ_1. Therefore, $(\prod_{i=1}^{r} p_i, \chi(1)) = 1$.

Now set $p = p_1$. By [**4**, Corollary 2.7], $\chi = \Phi\lambda$, where Φ and λ are irreducible, primitive π-special and π'-special characters. Since $\chi(1)$ is a π-number, λ is linear. As in the previous paragraph, $\beta \circ \Phi$ and $\beta \circ \lambda$ are π-special and π'-special. Thus, $\Phi\lambda = \chi = \beta \circ \chi = (\beta \circ \Phi)(\beta \circ \lambda)$ and [**4**, Theorem 2.2] imply that $\lambda = \beta \circ \lambda$. Since β fixes λ, β fixes all powers of λ. In particular, β fixes λ_p and $\lambda_{p'}$, where λ_p is the p-part of λ and $\lambda_{p'}$ is the p'-part. It is clear that $f(\lambda_p)$ is a p-number and $f(\lambda_{p'})$ is a p'-number. Hence, $\theta_i \circ \lambda_p = \lambda_p$ for $i = 2, \ldots, r$ and $\theta_1 \circ \lambda_{p'} = \lambda_{p'}$. Since $G(\Omega/Q)$ is abelian, $\lambda_p = \beta \circ \lambda_p = \theta_1 \circ \lambda_p$. By [**3**, Proposition 6.3(a)], $f(\Phi)$ is a π-number. Thus, $\theta_1 \circ \Phi = \Phi$ and $\theta_1 \circ \chi = (\theta_1 \circ \Phi)(\theta_1 \circ \lambda_p)(\theta_1 \circ \lambda_{p'}) = \Phi\lambda_p\lambda_{p'} = \chi$. Again this is a contradiction. Therefore, no product of an odd number of distinct θ_i leaves χ invariant. Hence, by [**2**, Theorem 2], there is an element $g \in G$ such that $\chi(g) \notin \Omega_i$ for any $i = 1, \ldots, n$. Thus, $f(\chi) \mid |\langle g \rangle|$.

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