ON A QUESTION OF FEIT
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ABSTRACT. The following theorem is proved: Assume $\chi$ is an irreducible complex character of the finite group $G$ and $G$ is $\pi$-solvable where $\pi$ is the set of prime divisors of $\chi(1)$. Then $G$ contains an element of order $f(\chi)$.

Introduction. All groups in this paper are finite. The question referred to in the title is the following: Let $\chi$ be an irreducible complex character of a finite group $G$ and let $f(\chi)$ be the smallest positive integer such that $\{\chi(g) \mid g \in G\} \subseteq \mathbb{Q}(\alpha)$ where $\alpha$ is a primitive $f(\chi)$th root of unity. Does $G$ contain an element of order $f(\chi)$?

By using factorizations of quasi-primitive irreducible characters into products of characters we obtain the following result:

THEOREM. Assume $\chi$ is an irreducible complex character of a finite group $G$ and $G$ is $\pi$-solvable where $\pi$ is the set of primes dividing $\chi(1)$. Then $G$ contains an element of order $f(\chi)$.

This theorem yields the following Corollary which was proved independently by Amit and Chillag [1].

COROLLARY. If $\chi$ is an irreducible character of a solvable group $G$, then $G$ contains an element of order $f(\chi)$.

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1. We introduce some definitions and notation. Let $\sigma$ be a nonempty set of primes and $h$ be an element in a group $G$, $h$ is a $\sigma$-element if $\langle h \rangle$ is a $\sigma$-group. Any $g \in G$ may be written uniquely as $g = g_{\sigma}g_{\sigma'}$, where $g_{\sigma}$ is a $\sigma$-element, $g_{\sigma'}$ is a $\sigma'$-element and both $g_{\sigma}$ and $g_{\sigma'}$ are powers of $g$. The elements $g_{\sigma}$ and $g_{\sigma'}$ are called the $\sigma$-part and $\sigma'$-part of $g$.

If $\Omega$ is a Galois extension of the field $\Omega_1$, $G(\Omega/\Omega_1)$ denotes the Galois group of $\Omega$ over $\Omega_1$.

If $\chi \in \text{Irr}(G)$ and $\sigma$ is a set of primes, then $\chi$ is $\sigma$-special provided that $\chi(1)$ is a $\sigma$-number and that for all subnormal subgroups $S$ of $G$ and all irreducible constituents $\theta$ of $\chi_S$, the determinantal order $O(\theta)$ is a $\sigma$-number.

PROOF OF THEOREM. The result is clear for linear characters so by induction on $\chi(1)$ we assume that the theorem is true for irreducible characters $\Phi$ and groups $H$ satisfying the hypothesis if $\Phi(1) < \chi(1)$. If $\chi = \Phi^G$ where $\Phi \in \text{Irr}(H)$ and

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|H| < |G|, then \( f(\chi) = f(\Phi) \). Since \( \Phi(1) \) is a proper divisor of \( \chi(1) \), \( H \) contains an element of order \( f(\Phi) \) and the theorem follows. Thus, we may assume that \( \chi \) is primitive.

Let \( f(\chi) = \prod_{i=1}^{n} p_i^{a_i} \) where \( a_i \geq 1 \) and the \( p_i \) are distinct primes. Let \( p_i^r \) denote the order of a Sylow \( p_i \)-subgroup of \( G \) for \( i = 1, \ldots, n \). Then \( \Omega_i \) denotes the field of \( (|G|/p_i^{r-a_i+1}) \)-th roots of unity over \( Q \). Let \( \Omega \) be a field of \( |G| \)-th roots of unity over \( Q \). By the definition of \( f(\chi) \), we may choose a \( \theta_i \in G(\Omega/\Omega_i) \) such that \( \theta_i \) does not leave \( \chi \) invariant for \( i = 1, \ldots, n \). We first show that no product of an odd number of distinct elements in \( \{\theta_1, \ldots, \theta_n\} \) leaves \( \chi \) invariant. Assume otherwise; then since \( G(\Omega/Q) \) is abelian, we may choose notation so that \( \beta = \prod_{i=1}^{r} \theta_i \) leaves \( \chi \) invariant, for some \( r \geq 1 \).

Suppose \( p_i | \chi(1) \) for some \( i = 1, \ldots, r \). Since \( G(\Omega/Q) \) is abelian, we may assume \( p_i | \chi(1) \). By [4, Corollary 2.7], \( \chi = \chi_1 \Phi \) where \( \chi_1 \) and \( \Phi \) are primitive irreducible \( p_1 \)-special and \( p'_1 \)-special characters. \( G \) is \( p_1 \)-solvable, hence \( p'_1 \)-separable, and so \( f(\Phi) \) is a \( p'_1 \)-number by [3, Proposition 6.3(a)].

If \( \theta \in G(\Omega/Q) \) and \( \sigma \) is any set of primes, then it is direct to see that whenever \( \gamma \) is an irreducible \( \sigma \)-special character so is \( \theta \circ \gamma \). Hence \( \beta \circ \chi_1 \) and \( \beta \circ \Phi \) are \( p_1 \)-special and \( p'_1 \)-special. Now \( \chi_1 \Phi = \chi = \beta \circ \chi = (\beta \circ \chi_1)(\beta \circ \Phi) \) and [4, Theorem 2.2] yield \( \chi_1 = \beta \circ \chi_1 \). By [3, Proposition 6.3(a)], \( f(\chi_1) \) is a \( p_1 \)-number so \( \theta_i \circ \chi_1 = \chi_1 \) for \( i = 2, \ldots, r \). Since \( G(\Omega/Q) \) is abelian, it follows that \( \chi_1 = \beta \circ \chi_1 = \theta_i \circ \chi_1 \). Now \( f(\Phi) \) a \( p'_1 \)-number implies that \( \theta_i \circ \Phi = \Phi \). However, \( \theta_i \circ \chi = (\theta_i \circ \chi_1)(\theta_i \circ \Phi) = \chi_1 \Phi = \chi \) contradicts the choice of \( \theta_i \). Therefore, \( \prod_{i=1}^{r} \theta_i \circ \chi(1) = 1 \).

Now set \( p = p_1 \). By [4, Corollary 2.7], \( \chi = \Phi \lambda \), where \( \Phi \) and \( \lambda \) are irreducible, primitive \( \pi \)-special and \( \pi' \)-special characters. Since \( \chi(1) \) is a \( \pi \)-number, \( \lambda \) is linear. As in the previous paragraph, \( \beta \circ \Phi \) and \( \beta \circ \lambda \) are \( \pi \)-special and \( \pi' \)-special. Thus, \( \Phi \lambda = \chi = \beta \circ \chi = (\beta \circ \Phi)(\beta \circ \lambda) \) and [4, Theorem 2.2] imply that \( \lambda = \beta \circ \lambda \). Since \( \beta \) fixes \( \lambda \), \( \beta \) fixes all powers of \( \lambda \). In particular, \( \beta \) fixes \( \lambda_p \) and \( \lambda_{p'} \), where \( \lambda_p \) is the \( p \)-part of \( \lambda \) and \( \lambda_{p'} \) is the \( p' \)-part. It is clear that \( f(\lambda_p) \) is a \( p \)-number and \( f(\lambda_{p'}) \) is a \( p' \)-number. Hence, \( \theta_i \circ \lambda_p = \lambda_p \) for \( i = 2, \ldots, r \) and \( \theta_1 \circ \lambda_{p'} = \lambda_{p'} \). Since \( G(\Omega/Q) \) is abelian, \( \lambda_p = \beta \circ \lambda_p = \theta_1 \circ \lambda_p \). By [3, Proposition 6.3(a)], \( f(\Phi) \) is a \( \pi \)-number. Thus, \( \theta_1 \circ \Phi = \Phi \) and \( \theta_1 \circ \chi = (\theta_1 \circ \Phi)(\theta_1 \circ \lambda_p)(\theta_1 \circ \lambda_{p'}) = \Phi \lambda_p \lambda_{p'} = \chi \). Again this is a contradiction. Therefore, no product of an odd number of distinct \( \theta_i \) leaves \( \chi \) invariant. Hence, by [2, Theorem 2], there is an element \( g \in G \) such that \( \chi(g) \notin \Omega_i \) for any \( i = 1, \ldots, n \). Thus, \( f(\chi) \mid \langle g \rangle \).

REFERENCES


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