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FACTORIZATION OF MEASURES AND PERFECTION

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ABSTRACT. It is proved that a probability measure P defined on a countably generated measurable space $(Y, \mathcal{C})$ is perfect iff every probability measure on $\mathbf{R} \times Y$ having P as marginal can be factored. This result leads to a generalization of a theorem due to Blackwell and Maitra.

We characterize countably generated measurable spaces $(Y, \mathcal{C})$ which have the following property: For any measurable space $(X, \mathcal{A})$ and any probability measure Q on the product space $(X \times Y, \mathcal{A} \otimes \mathcal{C})$, Q can be *factored*, $Q = \tilde{Q} \times K$, which means that $\tilde{Q}$ is a probability measure on $(X, \mathcal{A})$ and $K: X \times \mathcal{C} \to [0,1]$ is a transition probability (i.e. $K(\cdot, C)$ is $\mathcal{A}$-measurable for every $C \in \mathcal{C}$ and $K(x, \cdot)$ is a probability measure on $\mathcal{C}$ for every $x \in X$) such that $Q(A \times C) = \int_A K(x, C)\tilde{Q}(dx)$ holds for all $A \in \mathcal{A}$ and $C \in \mathcal{C}$. For the special case of a separable metric space Y equipped with its Borel σ-algebra, we obtain the characterization of absolutely measurable separable metric spaces given by Blackwell and Maitra in [**1**].

In the sequel we shall use the following notation. If Y is a topological space, then $\mathcal{B}(Y)$ denotes the Borel σ-algebra of Y. In particular, we denote by $\mathcal{B}$ the Borel σ-algebra of the real line $\mathbf{R}$ with the Euclidean topology. If $(X, \mathcal{A})$ and $(Y, \mathcal{C})$ are measurable spaces, $f: X \to Y$ is $\mathcal{A}, \mathcal{C}$-measurable and μ is a measure on $\mathcal{A}$, then μ^f denotes the image measure of μ under f. A probability measure P defined on a measurable space $(Y, \mathcal{C})$ is said to be *perfect* if for every $\mathcal{C}$-measurable real-valued function f on Y, there exists a set $B \in \mathcal{B}$ such that $B \subset f(Y)$ and $P(f^{-1}(B)) = 1$. Several other characterizations of perfect measures are given in [**5**].

We can now prove the main result of this note. Observe that the proof of our implication (3) $\Rightarrow$ (1) is a modification of the proof of the implication (c) $\Rightarrow$ (a) in [**1**].

THEOREM. *Let $(Y, \mathcal{C}, P)$ be a probability space. Then the following three statements are equivalent:*

(1) *P is perfect.*

(2) *For any measurable space $(X, \mathcal{A})$, any countably generated sub-σ-algebra $\mathcal{C}_0$ of $\mathcal{C}$ and any probability measure Q on $(X \times Y, \mathcal{A} \otimes \mathcal{C}_0)$ satisfying $Q(X \times C) = P(C)$ for all $C \in \mathcal{C}_0$, Q can be factored.*

(3) *For any countably generated sub-σ-algebra $\mathcal{C}_0$ of $\mathcal{C}$ and any probability measure Q on $(\mathbf{R} \times Y, \mathcal{B} \otimes \mathcal{C}_0)$ satisfying $Q(\mathbf{R} \times C) = P(C)$ for all $C \in \mathcal{C}_0$, Q can be factored.*

PROOF. (1) $\Rightarrow$ (2) Let $(X, \mathcal{A})$ be a measurable space, $\mathcal{C}_0$ a countably generated sub-σ-algebra of $\mathcal{C}$ and Q a probability measure on $\mathcal{A} \otimes \mathcal{C}_0$ satisfying $Q(X \times C) = P(C)$ for all $C \in \mathcal{C}_0$. Denote by π_1 $[\pi_2]$ the projection of $X \times Y$ onto X $[Y]$. By

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[**5**, Theorem 3], the image measure $Q^{\pi_2} = P|\mathcal{C}_0$ is compact. Thus, by [**2**, 5.3.16], there exists a regular conditional probability K of π_2 given π_1, i.e. K is a transition probability on $X \times \mathcal{C}_0$ such that

$$Q(A \times C) = Q(\pi_1^{-1}(A) \cap \pi_2^{-1}(C)) = \int_A K(x, C) Q^{\pi_1}(dx)$$

holds for all $A \in \mathcal{A}$ and $C \in \mathcal{C}_0$. Thus Q can be factored.

(2) $\Rightarrow$ (3) Trivial.

(3) $\Rightarrow$ (1) Let $f: Y \to \mathbf{R}$ be $\mathcal{C}$-measurable. Put $Z := f(Y)$ and denote by $\tilde{\mathcal{B}}$ the smallest σ-algebra in $\mathbf{R}$ containing $\mathcal{B} \cup \{Z\}$. Extend P^f to a measure $\tilde{P}$ on $\tilde{\mathcal{B}}$ by setting $\tilde{P}((B_1 \cap Z) + (B_2 - Z)) := P^f(B_1)$ for $B_1, B_2 \in \mathcal{B}$. Next define a probability Q_1 on $(\mathbf{R}^2, \mathcal{B} \otimes \tilde{\mathcal{B}})$ by $Q_1(B \times \tilde{B}) := \tilde{P}(B \cap \tilde{B})$, $B \in \mathcal{B}$, $\tilde{B} \in \tilde{\mathcal{B}}$. $\mathcal{C}_0 := f^{-1}(\mathcal{B})$ is a countably generated sub-σ-algebra of $\mathcal{C}$. For $B \in \mathcal{B}$ and $C_0 \in \mathcal{C}_0$, say $C_0 = f^{-1}(B_0)$ with $B_0 \in \mathcal{B}$, put $Q(B \times C_0) := Q_1(B \times (B_0 \cap Z))$. Then Q is a well-defined probability on $\mathcal{B} \otimes \mathcal{C}_0$ satisfying

$$Q(\mathbf{R} \times C_0) = Q_1(\mathbf{R} \times (B_0 \cap Z)) = \tilde{P}(B_0 \cap Z) = P^f(B_0) = P(C_0) \quad \text{for all } C_0 \in \mathcal{C}_0.$$

By (3), Q can be factored: $Q = \tilde{Q} \times K$, where $\tilde{Q}$ is a probability on $(\mathbf{R}, \mathcal{B})$ and K is a transition probability on $\mathbf{R} \times \mathcal{C}_0$. Then $K'(x, \tilde{B}) := K(x, f^{-1}(\tilde{B} \cap Z))$, $x \in \mathbf{R}$, $\tilde{B} \in \tilde{\mathcal{B}}$, defines a transition probability on $\mathbf{R} \times \tilde{\mathcal{B}}$, and we obtain for $B \in \mathcal{B}$ and $\tilde{B} \in \tilde{\mathcal{B}}$,

$$\begin{aligned}\tilde{P}(B \cap \tilde{B}) &= Q_1(B \times \tilde{B}) = Q_1(B \times (\tilde{B} \cap Z)) = Q(B \times f^{-1}(\tilde{B} \cap Z)) \\ &= \int_B K(x, f^{-1}(\tilde{B} \cap Z)) \tilde{Q}(dx) = \int_B K'(x, \tilde{B}) \tilde{Q}(dx),\end{aligned}$$

i.e.

$$(*) \qquad \tilde{P}(B \cap \tilde{B}) = Q_1(B \times \tilde{B}) = \int_B K'(x, \tilde{B}) \tilde{Q}(dx) \quad \text{for } B \in \mathcal{B},\ \tilde{B} \in \tilde{\mathcal{B}}.$$

Setting $\tilde{B} = Z$ in $(*)$, we get $P^f(B) = \tilde{Q}(B)$ for $B \in \mathcal{B}$, so $P^f = \tilde{Q}$. Setting $B = \tilde{B}$ in $(*)$, we obtain $P^f(B) = \int_B K'(x, B) P^f(dx)$ or

$$(**) \qquad \int_B (1 - K'(x, B)) P^f(dx) = 0 \quad \text{for all } B \in \mathcal{B}.$$

Let $\mathcal{E} = \{B_1, B_2, \ldots\}$ be a countable algebra generating $\mathcal{B}$. In view of $(**)$, we can find, for every $n \in N$, a set $N_n \in \mathcal{B}$ such that $P^f(N_n) = 0$ and $1_{B_n}(x) \cdot (1 - K'(x, B_n)) = 0$ for $x \in \mathbf{R} - N_n$. Put $\tilde{N} := \bigcup_{n \in N} N_n$ and $\mathcal{M} := \{B \in \mathcal{B}: 1_B(x) \cdot (1 - K'(x, B)) = 0 \text{ for all } x \in \mathbf{R} - \tilde{N}\}$. Then $\mathcal{M}$ is a monotone class containing $\mathcal{E}$. This implies $\mathcal{M} = \mathcal{B}$ and hence $1_B(x) \cdot (1 - K'(x, B)) = 0$ for all $B \in \mathcal{B}$ and all $x \in \mathbf{R} - \tilde{N}$. In particular, we get $K'(x, \{x\}) = 1$ for all $x \in \mathbf{R} - \tilde{N}$. By construction of K', we also have $K'(x, Z) = 1$ for all $x \in \mathbf{R}$. It follows that $\mathbf{R} - \tilde{N} \subset Z$ which together with $P^f(\mathbf{R} - \tilde{N}) = 1$ implies (1).

COROLLARY 1. *Let $(Y, \mathcal{C}, P)$ be a probability space where $\mathcal{C}$ is countably generated. Then the following three statements are equivalent:*

(1) P is perfect.

(2′) For any measurable space $(X, \mathcal{A})$ and any probability measure Q on $(X \times Y, \mathcal{A} \otimes \mathcal{C})$ satisfying $Q(X \times C) = P(C)$ for all $C \in \mathcal{C}$, Q can be factored.

(3′) *Every probability measure* Q *on* $(\mathbf{R} \times Y, \mathcal{B} \otimes \mathcal{C})$ *satisfying* $Q(\mathbf{R} \times C) = P(C)$ *for all* $C \in \mathcal{C}$ *can be factored.*

PROOF. In view of the Theorem, it suffices to prove the implications $(2) \Rightarrow (2')$, $(2') \Rightarrow (3')$ and $(3') \Rightarrow (3)$. Only the latter one is nontrivial.

Let $\mathcal{C}_0$ be a countably generated sub-σ-algebra of $\mathcal{C}$, and let Q be a probability on $\mathcal{B} \otimes \mathcal{C}_0$ satisfying $Q(\mathbf{R} \times C) = P(C)$ for all $C \in \mathcal{C}_0$. By means of a Hahn-Banach argument, combined with [**3**, 1(i)] (which can be applied since the marginal measure $B \in \mathcal{B} \to Q(B \times Y)$ is Radon and hence compact), Q can be extended to a probability measure $\tilde{Q}$ on $\mathcal{B} \otimes \mathcal{C}$ satisfying $\tilde{Q}(\mathbf{R} \times C) = P(C)$ for all $C \in \mathcal{C}$ (cf. the proof of 2.3 in [**4**]). Since, by (3′), $\tilde{Q}$ can be factored, so can Q. This proves $(3') \Rightarrow (3)$.

COROLLARY 2. *Let* $(Y, \mathcal{C})$ *be a countably generated measurable space. Then the following three statements are equivalent*:

(4) *Every probability measure on* $\mathcal{C}$ *is perfect.*

(5) *For any measurable space* $(X, \mathcal{A})$ *and any probability measure* Q *on* $(X \times Y, \mathcal{A} \otimes \mathcal{C})$, Q *can be factored.*

(6) *Every probability measure on* $(\mathbf{R} \times Y, \mathcal{B} \otimes \mathcal{C})$ *can be factored.*

The theorem of Blackwell and Maitra [**1**] is now an immediate consequence of Corollary 2 and the following proposition.

PROPOSITION. *A separable metric space* Y *is absolutely measurable (i.e., if* $\tilde{Y}$ *is a metric completion of* Y *and* μ *is a probability measure on* $\mathcal{B}(\tilde{Y})$, *then* Y *is* μ*-measurable) iff every probability measure on* $\mathcal{B}(Y)$ *is perfect.*

PROOF. According to [**5**, Theorem 11], the perfect probability measures on $\mathcal{B}(Y)$ are exactly the Radon probabilities on $\mathcal{B}(Y)$. On the other hand, any metric completion $\tilde{Y}$ of Y is Polish and hence a Radon space (cf. [**6**, p. 122]). Thus our claim follows from Propositions 8 and 9 in [**6**, pp. 118–119].

REMARK. Using the methods of Pachl (cf. [**4**, pp. 159–161]) one can even show that every probability space $(Y, \mathcal{C}, P)$ which satisfies condition (2′) of Corollary 1 is compact (and hence perfect). On the other hand, the complete Lebesgue measure on the unit interval, is an example of a compact probability P that does not satisfy (2′).

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