FACTORIZATION OF MEASURES AND PERFECTION
WOLFGANG ADAMSKI

ABSTRACT. It is proved that a probability measure $P$ defined on a countably generated measurable space $(Y, \mathcal{C})$ is perfect iff every probability measure on $\mathbb{R} \times Y$ having $P$ as marginal can be factored. This result leads to a generalization of a theorem due to Blackwell and Maitra.

We characterize countably generated measurable spaces $(Y, \mathcal{C})$ which have the following property: For any measurable space $(X, \mathcal{A})$ and any probability measure $Q$ on the product space $(X \times Y, \mathcal{A} \otimes \mathcal{C})$, $Q$ can be factored, $Q = \tilde{Q} \times K$, which means that $\tilde{Q}$ is a probability measure on $(X, \mathcal{A})$ and $K: X \times \mathcal{C} \to [0,1]$ is a transition probability (i.e. $K(\cdot, C)$ is $\mathcal{A}$-measurable for every $C \in \mathcal{C}$ and $K(x, \cdot)$ is a probability measure on $\mathcal{C}$ for every $x \in X$) such that $Q(A \times C) = \int_A K(x, C) \tilde{Q}(dx)$ holds for all $A \in \mathcal{A}$ and $C \in \mathcal{C}$. For the special case of a separable metric space $Y$ equipped with its Borel $\sigma$-algebra, we obtain the characterization of absolutely measurable separable metric spaces given by Blackwell and Maitra in [1].

In the sequel we shall use the following notation. If $Y$ is a topological space, then $\mathcal{B}(Y)$ denotes the Borel $\sigma$-algebra of $Y$. In particular, we denote by $\mathcal{B}$ the Borel $\sigma$-algebra of the real line $\mathbb{R}$ with the Euclidean topology. If $(X, \mathcal{A})$ and $(Y, \mathcal{C})$ are measurable spaces, $f: X \to Y$ is $\mathcal{A}, \mathcal{C}$-measurable and $\mu$ is a measure on $\mathcal{A}$, then $\mu^f$ denotes the image measure of $\mu$ under $f$. A probability measure $P$ defined on a measurable space $(Y, \mathcal{C})$ is said to be perfect if for every $\mathcal{C}$-measurable real-valued function $f$ on $Y$, there exists a set $B \in \mathcal{B}$ such that $B \subset f(Y)$ and $P(f^{-1}(B)) = 1$. Several other characterizations of perfect measures are given in [5].

We can now prove the main result of this note. Observe that the proof of our implication $(3) \Rightarrow (1)$ is a modification of the proof of the implication $(c) \Rightarrow (a)$ in [1].

THEOREM. Let $(Y, \mathcal{C}, P)$ be a probability space. Then the following three statements are equivalent:

(1) $P$ is perfect.

(2) For any measurable space $(X, \mathcal{A})$, any countably generated sub-$\sigma$-algebra $\mathcal{C}_0$ of $\mathcal{C}$ and any probability measure $Q$ on $(X \times Y, \mathcal{A} \otimes \mathcal{C}_0)$ satisfying $Q(X \times C) = P(C)$ for all $C \in \mathcal{C}_0$, $Q$ can be factored.

(3) For any countably generated sub-$\sigma$-algebra $\mathcal{C}_0$ of $\mathcal{C}$ and any probability measure $Q$ on $(\mathbb{R} \times Y, \mathcal{B} \otimes \mathcal{C}_0)$ satisfying $Q(\mathbb{R} \times C) = P(C)$ for all $C \in \mathcal{C}_0$, $Q$ can be factored.

PROOF. (1) $\Rightarrow$ (2) Let $(X, \mathcal{A})$ be a measurable space, $\mathcal{C}_0$ a countably generated sub-$\sigma$-algebra of $\mathcal{C}$ and $Q$ a probability measure on $\mathcal{A} \otimes \mathcal{C}_0$ satisfying $Q(X \times C) = P(C)$ for all $C \in \mathcal{C}_0$. Denote by $\pi_1 [\pi_2]$ the projection of $X \times Y$ onto $X$ [$Y$]. By
FACTORIZATION OF MEASURES AND PERFECTION

[5, Theorem 3], the image measure $Q^\pi_2 = P|\mathcal{C}_0$ is compact. Thus, by [2, 5.3.16], there exists a regular conditional probability $K$ of $\pi_2$ given $\pi_1$, i.e. $K$ is a transition probability on $X \times \mathcal{C}_0$ such that

$$Q(A \times C) = Q(\pi_1^{-1}(A) \cap \pi_2^{-1}(C)) = \int_A K(x,C)Q^\pi_1(dx)$$

holds for all $A \in \mathcal{A}$ and $C \in \mathcal{C}_0$. Thus $Q$ can be factored.

(2) $\Rightarrow$ (3) Trivial.

(3) $\Rightarrow$ (1) Let $f: Y \to \mathbb{R}$ be $\mathcal{C}$-measurable. Put $Z := f(Y)$ and denote by $\mathcal{B}$ the smallest $\sigma$-algebra in $\mathbb{R}$ containing $\mathcal{B} \cup \{Z\}$. Extend $P^f$ to a measure $\tilde{P}$ on $\mathcal{B}$ by setting $\tilde{P}((B_1 \cap Z) + (B_2 - Z)) := P^f(B_1)$ for $B_1, B_2 \in \mathcal{B}$. Next define a probability $Q_1$ on $(\mathbb{R}^2, \mathcal{B} \otimes \tilde{\mathcal{B}})$ by $Q_1(B \times \tilde{B}) := \tilde{P}(B \cap \tilde{B})$, $B \in \mathcal{B}$, $\tilde{B} \in \tilde{\mathcal{B}}$. $\mathcal{C}_0 := f^{-1}(\mathcal{B})$ is a countably generated sub-$\sigma$-algebra of $\mathcal{C}$. For $B \in \mathcal{B}$ and $C_0 \in \mathcal{C}_0$, say $C_0 = f^{-1}(B_0)$ with $B_0 \in \mathcal{B}$, put $Q(B \times C_0) := Q_1(B \times (B_0 \cap Z))$. Then $Q$ is a well-defined probability on $\mathcal{B} \otimes \mathcal{C}_0$ satisfying

$$Q(R \times C_0) = Q_1(R \times (B_0 \cap Z)) = \tilde{P}(B_0 \cap Z) = P^f(B_0) = P(C_0) \quad \text{for all } C_0 \in \mathcal{C}_0.$$ 

By (3), $Q$ can be factored: $Q = \tilde{Q} \times K$, where $\tilde{Q}$ is a probability on $(\mathbb{R}, \mathcal{B})$ and $K$ is a transition probability on $\mathbb{R} \times \mathcal{C}_0$. Then $K'(x, \tilde{B}) := K(x, f^{-1}(\tilde{B} \cap Z))$, $x \in \mathbb{R}$, $\tilde{B} \in \tilde{\mathcal{B}}$, defines a transition probability on $\mathbb{R} \times \tilde{\mathcal{B}}$, and we obtain for $B \in \mathcal{B}$ and $\tilde{B} \in \tilde{\mathcal{B}}$:

$$\tilde{P}(B \cap \tilde{B}) = Q_1(B \times \tilde{B}) = Q_1(B \times (\tilde{B} \cap Z)) = Q(B \times f^{-1}(\tilde{B} \cap Z))$$

$$= \int_B K(x, f^{-1}(\tilde{B} \cap Z))Q(x)dx = \int_B K'(x, \tilde{B})Q(x),$$

i.e.

$$(*) \quad \tilde{P}(B \cap \tilde{B}) = \tilde{Q}_1(B \times \tilde{B}) := \int_B K'(x, \tilde{B})Q(x)dx \quad \text{for } B \in \mathcal{B}, \tilde{B} \in \tilde{\mathcal{B}}.$$

Setting $\tilde{B} = Z$ in $(*)$, we get $P^f(B) = \tilde{Q}(B)$ for $B \in \mathcal{B}$, so $P^f = \tilde{Q}$. Setting $B = \tilde{B}$ in $(*)$, we obtain $P^f(B) = \int_B K'(x, B)P^f(dx)$ or

$$(**) \quad \int_B (1 - K'(x, B))P^f(dx) = 0 \quad \text{for all } B \in \mathcal{B}.$$

Let $\mathcal{E} = \{B_1, B_2, \ldots\}$ be a countable algebra generating $\mathcal{B}$. In view of $(**)$, we can find, for every $n \in N$, a set $N_n \in \mathcal{B}$ such that $P^f(N_n) = 0$ and $1_{B_n}(x) \cdot (1 - K'(x, B_n)) = 0$ for $x \in \mathbb{R} - N_n$. Put $\tilde{N} := \bigcup_{n \in N} N_n$ and $\mathcal{M} := \{B \in \mathcal{B}: 1_B(x) \cdot (1 - K'(x, B)) = 0 \text{ for all } x \in \mathbb{R} - \tilde{N}\}$. Then $\mathcal{M}$ is a monotone class containing $\mathcal{E}$. This implies $\mathcal{M} = \mathcal{B}$ and hence $1_B(x) \cdot (1 - K'(x, B)) = 0$ for all $B \in \mathcal{B}$ and all $x \in \mathbb{R} - \tilde{N}$. In particular, we get $K'(x, \{x\}) = 1$ for all $x \in \mathbb{R} - \tilde{N}$. By construction of $K'$, we also have $K'(x, Z) = 1$ for all $x \in \mathbb{R}$. It follows that $\mathbb{R} - \tilde{N} \subset Z$ which together with $P^f(\mathbb{R} - \tilde{N}) = 1$ implies (1).

**COROLLARY 1.** Let $(Y, \mathcal{C}, P)$ be a probability space where $\mathcal{C}$ is countably generated. Then the following three statements are equivalent:

(1) $P$ is perfect.

(2') For any measurable space $(X, \mathcal{A})$ and any probability measure $Q$ on $(X \times Y, \mathcal{A} \otimes \mathcal{C})$ satisfying $Q(X \times C) = P(C)$ for all $C \in \mathcal{C}$, $Q$ can be factored.
Every probability measure \( Q \) on \((R \times Y, \mathcal{B} \otimes \mathcal{C})\) satisfying \( Q(R \times C) = P(C) \) for all \( C \in \mathcal{C} \) can be factored.

**Proof.** In view of the Theorem, it suffices to prove the implications \((2) \Rightarrow (2')\), \((2') \Rightarrow (3')\) and \((3') \Rightarrow (3)\). Only the latter one is nontrivial.

Let \( \mathcal{C}_0 \) be a countably generated sub-\(\sigma\)-algebra of \( \mathcal{C} \), and let \( Q \) be a probability measure on \( \mathcal{B} \otimes \mathcal{C}_0 \) satisfying \( Q(R \times C) = P(C) \) for all \( C \in \mathcal{C}_0 \). By means of a Hahn-Banach argument, combined with [3, 1(i)] (which can be applied since the marginal measure \( B \in \mathcal{B} \mapsto Q(B \times Y) \) is Radon and hence compact), \( Q \) can be extended to a probability measure \( \tilde{Q} \) on \( \mathcal{B} \otimes \mathcal{C} \) satisfying \( \tilde{Q}(R \times C) = P(C) \) for all \( C \in \mathcal{C} \) (cf. the proof of 2.3 in [4]). Since, by \((3')\), \( \tilde{Q} \) can be factored, so can \( Q \). This proves \((3') \Rightarrow (3)\).

**Corollary 2.** Let \((Y, \mathcal{C})\) be a countably generated measurable space. Then the following three statements are equivalent:

1. Every probability measure on \( \mathcal{C} \) is perfect.
2. For any measurable space \((X, \mathcal{A})\) and any probability measure \( Q \) on \((X \times Y, \mathcal{A} \otimes \mathcal{C})\), \( Q \) can be factored.
3. Every probability measure on \((R \times Y, \mathcal{B} \otimes \mathcal{C})\) can be factored.

The theorem of Blackwell and Maitra [1] is now an immediate consequence of Corollary 2 and the following proposition.

**Proposition.** A separable metric space \( Y \) is absolutely measurable (i.e., if \( \tilde{Y} \) is a metric completion of \( Y \) and \( \mu \) is a probability measure on \( \mathcal{B}(\tilde{Y}) \), then \( Y \) is \( \mu \)-measurable) iff every probability measure on \( \mathcal{B}(Y) \) is perfect.

**Proof.** According to [5, Theorem 11], the perfect probability measures on \( \mathcal{B}(Y) \) are exactly the Radon probabilities on \( \mathcal{B}(Y) \). On the other hand, any metric completion \( \tilde{Y} \) of \( Y \) is Polish and hence a Radon space (cf. [6, p. 122]). Thus our claim follows from Propositions 8 and 9 in [6, pp. 118–119].

**Remark.** Using the methods of Pachl (cf. [4, pp. 159–161]) one can even show that every probability space \((Y, \mathcal{C}, P)\) which satisfies condition \((2')\) of Corollary 1 is compact (and hence perfect). On the other hand, the complete Lebesgue measure on the unit interval, is an example of a compact probability \( P \) that does not satisfy \((2')\).

**References**


Matematichesches Institut, Universität München, Theresienstr. 39, D-8000 München 2, Federal Republic of Germany